Application of He's Variational Iteration Method to Nonlinear Integro-Differential Equations

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In this paper, an application of He's variational iteration method is applied to solve nonlinear integro-differential equations. Some examples are given to illustrate the effectiveness of the method. The results show that the method provides a straightforward and powerful mathematical tool for solving various nonlinear integro-differential equations.

Key words: He's Variational Iteration Method; Nonlinear Integro-Differential Equations.

1. Introduction

In recent years, some promising approximate analytical solutions are proposed, such as exp-function method [1], homotopy perturbation method [2 – 11], and variational iteration method (VIM) [12 – 17]. The variational iteration method is the most effective and convenient one for both weakly and strongly nonlinear equations. This method has been shown to effectively, easily, and accurately solve a large class of nonlinear problems with component converging rapidly to accurate solutions.

Avudainayagam and Vani [18] considered the application of wavelet bases in solving integro-differential equations. They introduced a new four-dimensional connection coefficient and an algorithm for its computation. They tested their algorithm by solving two simple pedagogic nonlinear integro-differential equations. El-Shaled [19] and Ghasemi et al. [20 – 22] applied He's homotopy perturbation method to integro-differential equations. Ghasemi et al. [21, 22] and Kajani et al. [23] applied the Wavelet-Galerkin method and the sine-cosine wavelet method to integro-differential equations. Also recently, Darania and Ebadian [24] applied the differential transform method to integro-differential equations.

In this paper, we propose VIM to solve the nonlinear integro-differential equations. The Volterra integro-differential equation is given by

$$u'(x) = \nu(x) + \int_0^x k(x,t,u(t),u'(t))\mathrm{d}t \tag{1}$$

and Fredholm type is given by

$$u'(x) = \nu(x) + \int_a^b k(x,t,u(t),u'(t))\mathrm{d}t. \tag{2}$$

It was Wang and He [25] who first applied the variational iteration method to integro-differential equations. Lately Saberi-Nadjafi [26] found the method is a highly promising method for solving the system of integro-differential equations. Also He [27, 28] gave new interpretations of the variational iteration method for solving integro-differential equations.

2. He's Variational Iteration Method

Now, to illustrate the basic concept of He's variational iteration method, we consider the following general nonlinear differential equation given in the form

$$Lu(t) + Nu(t) = g(t), \tag{3}$$

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is a known analytical function. We can construct a correction functional according to the variational method as:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\,(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi))\mathrm{d}\xi, \tag{4}$$

where λ is a general Lagrange multiplier, which can be idendified optimally via the variational theory, the subscript n denotes the nth approximation, and $\tilde{u}_n$ is considered as a restricted variation, namely $\delta\tilde{u}_n = 0$ [12].

In the following examples, we will illustrate the usefulness and effectiveness of the proposed technique.

3. Numerical Examples

This section contains six examples of Volterra and Fredholm nonlinear integro-differential equations.

Example 1. Consider the following nonlinear integro-differential equation:

$$u'(x) = 1 + \int_0^x u(t)u'(t)\,\mathrm{d}t \tag{5}$$

for $x \in [0,1]$ with the exact solution $u(x) = \sqrt{2}\tan\left(\frac{\sqrt{2}}{2}x\right)$.

Using He's variational iteration method, the correction functional can be written in the form

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left\{u_n{}'(s) - 1 - \int_0^s u_n(p)\,\mathrm{d}p\right\}\mathrm{d}s. \tag{6}$$

The stationary conditions

$$1 + \lambda = 0, \quad \lambda' = 0 \tag{7}$$

follow immediately. This in turn gives

$$\lambda = -1. \tag{8}$$

Substituting this value of the Lagrange multiplier, $\lambda = -1$, into the functional (6) gives the iteration formula

$$u_{n+1}(x) = u_n(x) - \int_0^x \left\{u_n{}'(s) - 1 - \int_0^s u_n(p)u_n{}'(p)\,\mathrm{d}p\right\}\mathrm{d}s. \tag{9}$$

By VIM, let $L(u) = u'(x) - v(x) = 0$, we can choose $u_0(x)$ from the equation

$$L(u)_0 = u_0{}'(x) - 1 = 0. \tag{10}$$

We can select $u_0(x) = x$ from (10). Using this selection in (9), we obtain the following successive approximations:

$$u_0(x) = x, \tag{11}$$

$$u_1(x) = x + \frac{x^3}{6}, \tag{12}$$

$$u_2(x) = x + \frac{x^3}{6} + \frac{x^5}{30} + \frac{x^7}{504}, \tag{13}$$

$$u_3(x) = x + \frac{x^3}{6} + \frac{x^5}{30} + \frac{17x^7}{2520} + \frac{19x^9}{22680} + \frac{67x^{11}}{831600} + \frac{x^{13}}{196560} + \frac{x^{15}}{7620480}, \tag{14}$$

$$\vdots$$

Table 1. Numerical results of Example 1.

x	Exact Solution	VIM-u_3	Absolute Error
0.0	0	0	0
0.1	0.1001670006	0.1001670007	1×10^{-9}
0.2	0.2013440870	0.2013440868	2×10^{-9}
0.3	0.3045825026	0.3045824920	1.06×10^{-8}
0.4	0.4110194227	0.4110192757	1.47×10^{-7}
0.5	0.5219305152	0.5219293796	1.13×10^{-6}
0.6	0.6387957040	0.6387895873	6.11×10^{-6}
0.7	0.7633858019	0.7633600137	2.57×10^{-5}
0.8	0.8978815369	0.8977903903	9.11×10^{-5}
0.9	1.045043135	1.044760768	2.82×10^{-4}
1.0	1.208460241	1.207669561	7.9×10^{-4}

The results and the corresponding absolute errors are presented in Table 1 (with third-order approximation (14)).

Table 1 shows that the numerical approximate solution has a high degree of accuracy. As we know, the more terms added to the approximate solution, the more accurate it will be. Although we only considered a third-order approximation, it achieves a high level of accuracy.

Example 2. Consider the following nonlinear integro-differential equation:

$$u'(x) = -\frac{1}{2} + \int_0^x u'^2(t)\,\mathrm{d}t \tag{15}$$

for $x \in [0,1]$ with the exact solution $u(x) = -\ln\left(\frac{1}{2}x + 1\right)$.

We can construct a variational iteration form for (15) in the form:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left\{u_n{}'(s) + \frac{1}{2} - \int_0^s u_n{}'^2(p)\,\mathrm{d}p\right\}\mathrm{d}s. \tag{16}$$

By VIM, let $L(u) = u'(x) - v(x) = 0$, we can choose $u_0(x)$ from the equation

$$L(u_0) = u_0{}'(x) + \frac{1}{2} = 0. \tag{17}$$

We can select $u_0(x) = -\frac{x}{2}$ from (17). Using this selection in (16), we obtain the following successive approximations:

$$u_0(x) = -\frac{x}{2}, \tag{18}$$

Table 2. Numerical results of Example 2.

x	Exact Solution	VIM-u_3	Absolute Error
0.0	0	0	0
0.1	−0.04879016417	−0,04879014498	1.91×10^{-8}
0.2	−0.09531017980	−0.09530961268	5.67×10^{-7}
0.3	−0.1397619424	−0.1397579563	3.98×10^{-6}
0.4	−0.1823215568	−0.1823059759	1.55×10^{-5}
0.5	−0.2231435513	−0.2230993543	4.41×10^{-5}
0.6	−0.2623642645	−0.2622618412	1.02×10^{-4}
0.7	−0.3001045925	−0.2998980358	2.06×10^{-4}
0.8	−0.3364722366	−0.3360958171	3.76×10^{-4}
0.9	−0.3715635564	−0.3709284685	6.35×10^{-4}
1.0	−0.4054651081	−0.4044565353	1×10^{-3}

$$u_1(x) = -\frac{x}{2}+\frac{x^2}{8}, \tag{19}$$

$$u_2(x) = -\frac{x}{2}+\frac{x^2}{8}-\frac{x^3}{24}+\frac{x^4}{192}, \tag{20}$$

$$u_3(x) = -\frac{x}{2}+\frac{x^2}{8}-\frac{x^3}{24}+\frac{x^4}{64}+\frac{x^5}{240}+\frac{x^6}{1152} - \frac{x^7}{8064}+\frac{x^8}{129024}, \tag{21}$$

$$\vdots$$

The results and the corresponding absolute errors are presented in Table 2 (with third-order approximation (21)).

Table 2 shows that the numerical approximate solution has a high degree of accuracy. As we know, the more terms added to the approximate solution, the more accurate it will be. Although we only considered a third-order approximation, it achieves a high level of accuracy.

Example 3. Consider the following second-order nonlinear integro-differential equation:

$$u''(x) = \mathrm{e}^x - x + \int_0^1 xtu(t)\,\mathrm{d}t, \tag{22}$$

with the initial conditions

$$u(0) = 1, \quad u'(0) = 1 \tag{23}$$

for $x \in [0,1]$ with the exact solution $u(x) = \mathrm{e}^x$.

Making $u_{n+1}(x)$ stationary with respect to $u_n(x)$, we can identify the Lagrange multiplier, which reads

$$\lambda = s - x. \tag{24}$$

So we can construct a variational iteration form for (22) in the form:

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-x)\Big\{u_n''(s) - \mathrm{e}^s + s - \int_0^1 spu(p)\,\mathrm{d}p\Big\}\mathrm{d}s. \tag{25}$$

We begin with

$$u_0(x) = \mathrm{e}^x(a+bx), \tag{26}$$

where a and b are unknown constants to be further determined.

By the iteration formulation (25), we have

$$u_1(x) = (a-1)+(a+b-1)x + \left(-\frac{1}{6}+\frac{1}{6}a-\frac{1}{3}b+\frac{1}{6}be\right)x^3+\mathrm{e}^x. \tag{27}$$

If the first-order approximate solution is enough, by the aid of the initial conditions (23), we can identify the unknown constants as

$$a = 1 \text{ and } b = 0. \tag{28}$$

So we obtain the following first-order approximate solution:

$$u(x) = \mathrm{e}^x \tag{29}$$

which is the exact solution of the problem.

Example 4. Now, we consider the following third-order nonlinear integro-differential equation:

$$u'''(x) = \sin x - x - \int_0^{\pi/2} xtu'(t)\,\mathrm{d}t, \tag{30}$$

with the initial conditions

$$u(0) = 1,\ u'(0) = 0, \text{ and } u''(0) = -1 \tag{31}$$

for $x \in [0,\pi/2]$ with the exact solution $u(x) = \cos x$.

Making $u_{n+1}(x)$ stationary with respect to $u_n(x)$, we can identify the Lagrange multiplier, which reads

$$\lambda = \frac{(s-x)^2}{2}. \tag{32}$$

So we can construct a variational iteration form for (30) in the form:

$$u_{n+1}(x) = u_n(x) - \int_0^x \frac{(s-x)^2}{2}\Big\{u_n'''(s) - \sin s + s + \int_0^{\pi/2} spu'(p)\,\mathrm{d}p\Big\}\mathrm{d}s. \tag{33}$$

We begin with

$$u_0(x) = a + bx + cx^2, \tag{34}$$

where a, b and c are unknown constants to be further determined.

By the iteration formulation (33), we have

$$u_1(x) = (a-1) + bx + \left(c + \frac{1}{2}\right)x^2 + \cos x. \quad (35)$$

If the first-order approximate solution is enough, by the aid of the initial conditions (31), we can identify the unknown constants as

$$a = 1, \quad b = 0, \quad \text{and} \quad c = -1/2. \quad (36)$$

So we obtain the following first-order approximate solution:

$$u(x) = \cos x \quad (37)$$

which is the exact solution of the problem.

Example 5. Finally, we consider the following fifth-order integro-differential equation:

$$u^{(v)}(x) - u'(x) = \int_{-1}^{1} u(t)\mathrm{d}t, \quad (38)$$

with initial conditions

$$u(0) = 0,\ u'(0) = 1,\ u''(0) = 0,\ u'''(0) = -1, \\ \text{and} \ \ u'^{v}(0) = 0 \quad (39)$$

for $x \in [-1, 1]$ with the exact solution $u(x) = \sin x$.

Making $u_{n+1}(x)$ stationary with respect to $u_n(x)$, we can identify the Lagrange multiplier, which reads

$$\lambda = \frac{(s-x)^4}{24}. \quad (40)$$

So we can construct a variational iteration form for (38) in the form:

$$u_{n+1}(x) = u_n(x) - \int_0^x \frac{(s-x)}{24}\left\{u_n{}^{(v)}(s) - u_n{}'(s) - \int_{-1}^{1} u(p)\mathrm{d}p\right\}\mathrm{d}s. \quad (41)$$

We begin with

$$u_0(x) = a + bx + cx^2 + dx^3 + ex^4, \quad (42)$$

where a, b, c, d, and e are unknown constants to be further determined.

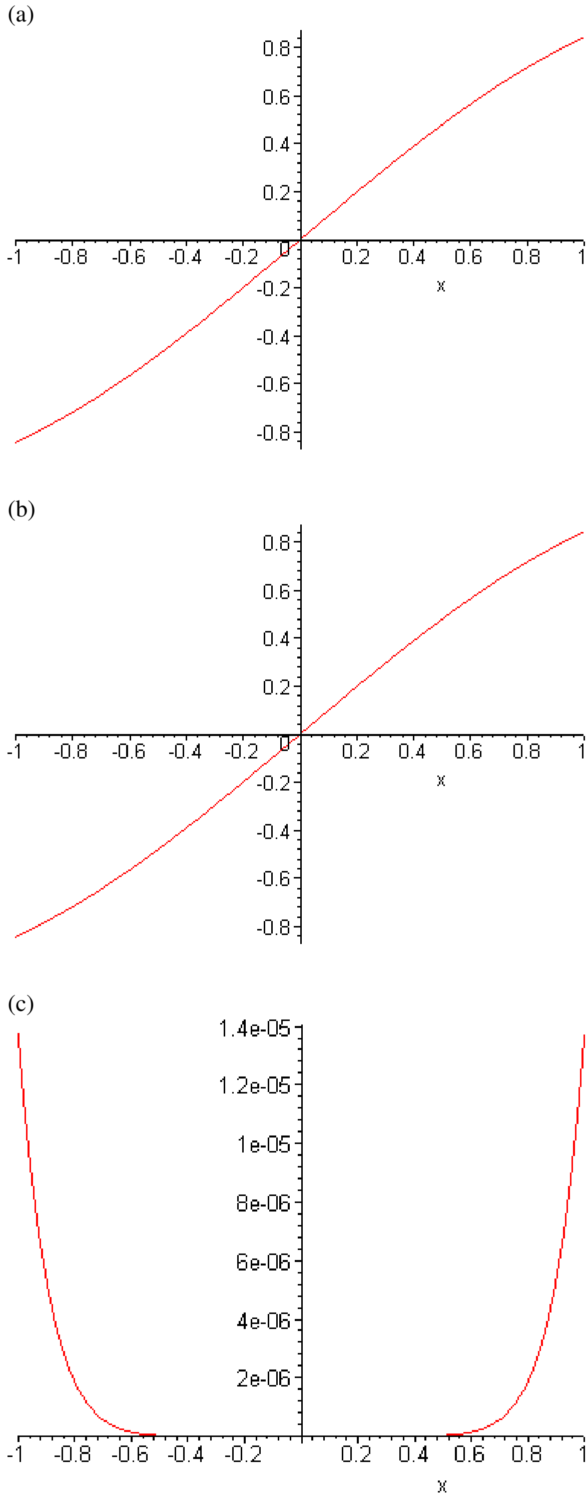

Fig. 1. (a) Exact Solution, (b) approximate Solution, (c) absolute error.

By the iteration formulation (41), we have

$$u_1(x) = a + bx + cx^2 + dx^3 + ex^4 + \left(\frac{a}{60} + \frac{b}{120} + \frac{c}{180}\right)x^5 + \left(\frac{a}{720} + \frac{c}{360}\right)x^6 + \left(\frac{d}{280} + \frac{b}{2520}\right)x^7 + \left(\frac{e}{240} + \frac{c}{6720}\right)x^8 + \frac{d}{15120}x^9 + \frac{e}{30240}x^{10}. \quad (43)$$

If the first-order approximate solution is enough, by the aid of the initial conditions (39), we can identify the unknown constants as

$$a = 0,\ b = 1,\ c = 0,\ d = -1/6,\ \text{and}\ e = 0. \quad (44)$$

So we obtain the following first-order approximate solution:

$$u_1(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \frac{x^9}{90720}. \quad (45)$$

The results and the corresponding absolute errors are presented in Figure 1 (with first-order approximation (45)). This figure shows that the numerical approximate solution has a high degree of accuracy. As we know, the more terms added to the approximate solution, the more accurate it will be. Although we only considered a first-order approximation, it achieves a high level of accuracy.

4. Conclusion

In this paper, we applied an application of He's variational iteration method for solving nonlinear integro-differential equations. The method is extremely simple, easy to use and is very accurate for solving nonlinear integro-differential equation. The solution obtained by VIM is valid for not only weakly nonlinear equations, but also strong ones. Also, the method is a powerful tool to search for solutions of various linear/nonlinear problems. This variational iteration method will become a much more interesting method to solve nonlinear integro-differential equation in science and engineering.

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