Blow Up of Solutions for a Viscoelastic System with Damping and Source Terms in $\mathbb{R}^n$

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This paper deals with a Cauchy problem for the coupled system of nonlinear viscoelastic equations with damping and source terms. We prove a new finite time blow-up result for compactly supported initial data with non-positive initial energy as well as positive initial energy by using the modified energy method and the compact support technique.

Key words: Blow Up; Viscoelastic Equation; Coupled System; Damping Term.

1. Introduction

In this paper, we consider the Cauchy problem of the following coupled system of nonlinear viscoelastic equations with damping and source terms:

$$
\begin{aligned}
& u_{tt}-\Delta u+\int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s+|u_t|^{m-2}u_t=f_1(u,v), \\
& \qquad (x,t)\in\mathbb{R}^n\times(0,\infty), \\
& v_{tt}-\Delta v+\int_0^t h(t-s)\Delta v(x,s)\mathrm{d}s+|v_t|^{r-2}v_t=f_2(u,v), \\
& \qquad (x,t)\in\mathbb{R}^n\times(0,\infty), \\
& u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\mathbb{R}^n, \\
& v(x,0)=v_0(x), \quad v_t(x,0)=v_1(x), \quad x\in\mathbb{R}^n,
\end{aligned}
\tag{1}
$$

where m, $r\geq 2$, and g,h, u_0, u_1, v_0, and v_1 are functions to be specified later. This type of problems arises naturally in the theory of viscoelasticity and describes the interaction of two scalar fields (see [1, 2]). The integral terms express the fact that the stress at any instant depends not only on the present value but on the entire past history of strains the material has undergone. In [3], the present author and Yu studied problem (1) for the case $m=r=2$ and proved a finite time blow-up result with vanishing initial energy. In the present work, we shall extend the result to the case m, $r\geq 2$ and for initial energy which may take positive values.

The motivation of our work is due to the initial boundary problem of the scalar equation

$$
\begin{aligned}
& u_{tt}-\Delta u+\int_0^t g(t-\tau)\Delta u(\tau)\mathrm{d}\tau+|u_t|^{m-2}u_t=|u|^{p-2}u, \\
& \qquad (x,t)\in\Omega\times(0,\infty), \\
& u(x,t)=0, \quad x\in\partial\Omega, \quad t\geq 0, \\
& u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\Omega,
\end{aligned}
\tag{2}
$$

where Ω is a bounded domain of $\mathbb{R}^n$ ($n\geq 1$) with a smooth boundary $\partial\Omega$, $p>2$, $m\geq 2$, and $g\colon\mathbb{R}^+\to\mathbb{R}^+$ is a positive non-increasing function. In the paper [4], Messaoudi showed, under suitable conditions on g, that solutions with negative initial energy blow up in finite time if $p>m$ while continue to exist if $m\geq p$. This result has been later pushed by the same author [5] to certain solutions with positive initial energy. For results of same nature, we refer the reader to [6 – 16].

In all above treatments the underlying domain is assumed to be bounded. The boundedness of the domain is essential because of the usage of the boundedness of the injection $L^p(\Omega)\subset L^q(\Omega)$ when $1\leq q\leq p$ (see Todorova [17, 18]). For (2) in $\mathbb{R}^n$, we mention the work of Kafini and Messaoudi [19], Levine et al. [20], Messaoudi [21], Sun and Wang [22], Tatar [23], Todorova [17, 18], and Zhou [24]. For example, Todorova [17] studied the Cauchy problem

$$
u_{tt}-\Delta u+|u_t|^{m-2}u_t=|u|^{p-2}u, \tag{3}
$$

$$(x,t) \in \mathbb{R}^n \times (0,\infty), \quad u(x,0) = u_0(x),$$
$$u_t(x,0) = u_1(x), \qquad x \in \mathbb{R}^n,$$
$$p \le 2(n-1)/(n-2), \text{ if } n > 2,$$

and showed that

- when $m \ge p$, (3) has a unique global solution,
- when $2 < m < p$ and $m > np/(n+p+1)$, the weak solution of (3) blows up in finite time for any compactly supported initial data with negative initial energy,
- when $2 < m < p$ and $m \le np/(n+p+1)$, the weak solution of (3) blows up in finite time for any compactly supported initial data with sufficiently negative initial energy and $\int_{\mathbb{R}^n} u_0 u_1 \mathrm{d}x \ge 0$.

Messaoudi [21] improved the above results by using a different functional so that the condition $\int_{\mathbb{R}^n} u_0 u_1 \mathrm{d}x \ge 0$ can be removed. Tatar [23] considered the Cauchy problem with a nonlinear dissipation of cubic convolution type and proved a finite-time blow-up result for initial energy which may take positive values. Recently, Kafini and Messaoudi [25] studied the coupled system (1) but without damping terms. By defining the functional

$$F(t) = \frac{1}{2}\int_{\mathbb{R}^n} [|u(x,t)|^2 + |v(x,t)|^2]\mathrm{d}x + \frac{1}{2}\beta(t+t_0)^2 \quad (4)$$

and using the classical concavity method, they proved that the solution blows up in finite time if the initial energy is negative. More recently, the same authors [19] considered the following Cauchy problem with a linear damping term ($m = 2$):

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s + |u_t|^{m-2}u_t = |u|^{p-2}u,$$
$$(x,t) \in \mathbb{R}^n \times (0,\infty),$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n. \quad (5)$$

Under suitable conditions on the initial data and the condition

$$\int_0^{+\infty} g(s)\mathrm{d}s < \frac{p-2}{p-3/2} \quad (6)$$

on the relaxation function, they proved a blow-up result with vanishing initial energy.

Motivated by the above mentioned researches, we intend to extend and improve the results in [19,25] by studying the problem (1) in this paper. We shall prove a blow-up result for a larger class of initial energy which may take positive values. Moreover, our result do not require $\int_{\mathbb{R}^n} u_0 u_1 \mathrm{d}x \ge 0$ if the initial energy is negative. We note that the method used in [19] cannot be applied to our problem since the damping terms are contained. For our purpose, we combine the method in [23] with the modified energy methods used in [4], where the case of a bounded domain with Dirichlet boundary condition was discussed. The lack of the injection $L^p(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ shall be compensated by the usage of the compact support technique.

This paper is organized as follows: In Section 2 we shall make some assumptions, give the local existence of solutions, and state the main results. In Section 3 we will prove the main result.

2. Preliminaries

In this section we present some assumptions, give the local existence of solutions, and state the main result. We use the standard Lebesgue space $L^p(\mathbb{R}^n)$ and the Sobolev space $H^1(\mathbb{R}^n)$ with their usual scalar product and norms.

We first make the following assumptions:

(G1) $g, h: \mathbb{R}_+ \to \mathbb{R}_+$ are non-increasing differentiable functions satisfying

$$1 - \int_0^\infty g(s)\,\mathrm{d}s = l > 0, \quad g(0) > 0,$$
$$1 - \int_0^\infty h(s)\,\mathrm{d}s = k > 0, \quad h(0) > 0.$$

(G2) There exists a function $I(u,v) \ge 0$ such that

$$\frac{\partial I}{\partial u} = f_1(u,v), \quad \frac{\partial I}{\partial v} = f_2(u,v),$$
$$uf_1(u,v) + vf_2(u,v) = pI(u,v).$$

(G3) There exist positive constants C_0, C_1, and $p > 2$ such that

$$C_0 \int_{\mathbb{R}^n} (|u|^p + |v|^p)\,\mathrm{d}x \le \int_{\mathbb{R}^n} I(u,v)\,\mathrm{d}x \le C_1 \int_{\mathbb{R}^n} (|u|^p + |v|^p)\,\mathrm{d}x.$$

(G4) There exists a constant $d > 0$ such that

$$|f_1(\xi,\varsigma)| \le d(|\xi|^{\gamma_1} + |\varsigma|^{\gamma_2}), \quad \forall(\xi,\varsigma) \in \mathbb{R}^2,$$
$$|f_2(\xi,\varsigma)| \le d(|\xi|^{\gamma_3} + |\varsigma|^{\gamma_4}), \quad \forall(\xi,\varsigma) \in \mathbb{R}^2,$$

where $\gamma_i \ge 1, \quad (n-2)\gamma_i \le n, \quad i = 1,2,3,4.$

Remark 2.1 (G1) is necessary to guarantee the hyperbolicity of the system (1). Condition (G4) is necessary for the existence of a local solution to (1). As an example of functions satisfying (G2) – (G4), we have

$$I(u,v) = \frac{1}{p}(2|uv|^{\frac{p}{2}} + |u+v|^p),$$
$$p \le 2(n-1)/(n-2) \text{ if } n > 2.$$

We introduce the 'modified' energy functional as in [25]:

$$\begin{aligned} E(t) := &\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|v_t\|_2^2 \\ &+ \frac{1}{2}\Big(1 - \int_0^t g(s)\mathrm{d}s\Big)\|\nabla u\|_2^2 \\ &+ \frac{1}{2}\Big(1 - \int_0^t h(s)\mathrm{d}s\Big)\|\nabla v\|_2^2 \\ &+ \frac{1}{2}(g \circ \nabla u) + \frac{1}{2}(h \circ \nabla v) \\ &- \int_{\mathbb{R}^n} I(u,v)\mathrm{d}x, \end{aligned} \tag{7}$$

where

$$\begin{aligned} (g \circ \nabla u)(t) &= \int_0^t g(t-\tau)\|\nabla u(t) - \nabla u(\tau)\|_2^2 \mathrm{d}\tau, \\ (h \circ \nabla v)(t) &= \int_0^t h(t-\tau)\|\nabla v(t) - \nabla v(\tau)\|_2^2 \mathrm{d}\tau. \end{aligned} \tag{8}$$

We now state, without a proof, a local existence result, which can be established by combining the arguments of [1], [17], and [26].

Theorem 2.2 Assume that (G1) and (G4) hold. Then for initial data $(u_0,\ u_1)$, $(v_0, v_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, with compact support $\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \cup \operatorname{supp} v_0 \cup \operatorname{supp} v_1 \subset B_R(0)$, problem (1) has a unique local solution

$$(u,v) \in [C([0,T); H^1(\mathbb{R}^n)]^2,$$
$$(u_t, v_t) \in [C([0,T); L^2(\mathbb{R}^n)) \cap L^m([0,T) \times \mathbb{R}^n)]^2$$

for T small enough.

Our main result reads as follows:

Theorem 2.3 Let $2 \le m,\ r < p$, and $p \le 2(n-1)/(n-2)$ if $n > 2$. Assume that (G1) – (G4) hold and that

$$\max\left\{\int_0^{+\infty} g(s)\mathrm{d}s, \int_0^{+\infty} h(s)\mathrm{d}s\right\} < \frac{p-2}{p-2+1/p}. \tag{9}$$

Assume further that $E(0) < 0$ or $E(0) \ge 0$ and $\int_{\mathbb{R}^n}(u_0u_1 + v_0v_1)\mathrm{d}x \ge 0$. Then for any $T > 0$ we can find initial data $(u_0,\ u_1)$, $(v_0, v_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, with compact support, such that the corresponding solution for the problem (1) blows up in finite time $t^* \le T$.

If we set $u = v$ and $m = r$, we get

Corollary 2.4 Let $2 \le m < p$, and $p \le 2(n-1)/(n-2)$ if $n > 2$. Suppose that g satisfies (G1) and

$$\int_0^{+\infty} g(s)\mathrm{d}s < \frac{p-2}{p-2+1/p}. \tag{10}$$

Assume further that $E(0) < 0$ or $E(0) \ge 0$ and $\int_{\mathbb{R}^n} u_0u_1\mathrm{d}x \ge 0$. Then for any $T > 0$ we can find initial data $(u_0,\ u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, with compact support, such that the solution for the problem (5) blows up in finite time $t^* \le T$.

Remark 2.5 We note here that the condition (10) is slightly weaker than (6), i. e., the one made in Kafini and Messaoudi [19], where the linear damping case ($m = 2$) was studied. Moreover, our method to deal with both the linear and nonlinear damping cases ($m > 2$) allows a larger class of initial energy which may take positive values.

Remark 2.6 Our discussion is applicable to the problem studied in [25], i. e., the case without damping terms. We note that a blow-up result was proved in [25] for only negative initial energy.

In order to prove our main result, we need the following lemma:

Lemma 2.7 [21] Suppose that $p \le 2(n-1)/(n-2)$ if $n > 2$, and $2 \le \alpha \le p$. Then there exists a positive constant C depending only on n and p such that

$$\|u\|_p^\alpha \le C(L)^{2/p-2/p^*}\left(\|\nabla u\|_2^2 + \|u\|_p^p\right) \tag{11}$$

for any $u \in H^1(\mathbb{R}^n)$, with $\operatorname{supp} u \subset B_L(0)$, where $p^* = 2n/(n-2)$ ($1/p^* = 0$ if $n = 2$).

3. Proof of Theorem 2.3

Assume by contradiction that the solution u is global. Multiplying the equations in (1) by u_t and v_t, respectively, and integrating over $\mathbb{R}^n$, we obtain (see [4])

$$E'(t) = -\left(\|u_t\|_m^m + \|v_t\|_r^r\right) + \frac{1}{2}(g' \circ \nabla u) + \frac{1}{2}(h' \circ \nabla v) - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}h(t)\|\nabla v\|_2^2 \leq 0.$$

Hence,

$$E(t) \leq E(0), \quad \text{for all } t \in [0,T].$$

We set

$$J(t) := -\int_0^t E(s)\mathrm{d}s + (\rho t + \omega)\int_{\mathbb{R}^n}(u_0^2(x) + v_0^2(x))\mathrm{d}x, \tag{12}$$

where ρ and ω are two positive constants to be chosen later. A differentiation of $J(t)$ implies that

$$\begin{aligned} J'(t) &= -E(t) + \rho\int_{\mathbb{R}^n}(u_0^2(x) + v_0^2(x))\mathrm{d}x \\ &\geq \rho\int_{\mathbb{R}^n}(u_0^2(x) + v_0^2(x))\mathrm{d}x - E(0). \end{aligned} \tag{13}$$

This combined with the choice of ρ satisfying

$$\rho\int_{\mathbb{R}^n}(u_0^2(x) + v_0^2(x))\,\mathrm{d}x - E(0) = J'(0) > 0,$$

imply that

$$J'(t) \geq J'(0) > 0, \quad \text{for all } t \in [0,T].$$

Furthermore, we have

$$\begin{aligned} J'(t) - J'(0) &= E(0) - E(t) = -\int_0^t E'(s)\mathrm{d}s \\ &\geq \int_0^t (\|u_t\|_m^m + \|v_t\|_r^r)\mathrm{d}s. \end{aligned} \tag{14}$$

We now define

$$K(t) := J^{1-\gamma}(t) + \varepsilon\int_0^t\int_{\mathbb{R}^n}(uu_t + vv_t)\mathrm{d}x\,\mathrm{d}s, \tag{15}$$

where $\varepsilon > 0$ is small to be chosen later, and

$$0 < \gamma \leq \min\left\{\frac{p-2}{2p}, \frac{p-m}{p(m-1)}, \frac{p-r}{p(r-1)}\right\}. \tag{16}$$

By taking a derivative of (15), we have

$$\begin{aligned} K'(t) &= (1-\gamma)J^{-\gamma}(t)J'(t) + \varepsilon\int_{\mathbb{R}^n}(uu_t + vv_t)\mathrm{d}x = \\ &(1-\gamma)J^{-\gamma}(t)J'(t) + \varepsilon\int_{\mathbb{R}^n}(u_0u_1 + v_0v_1)\mathrm{d}x \\ &+ \varepsilon\int_0^t\int_{\mathbb{R}^n}(u_t^2 + v_t^2)\mathrm{d}x\mathrm{d}s \\ &+ \varepsilon\int_0^t\int_{\mathbb{R}^n}(uu_{tt} + vv_{tt})\mathrm{d}x\mathrm{d}s. \end{aligned} \tag{17}$$

Multiplying the equations in (1) by u and v, respectively, and integrating over $\mathbb{R}^n \times (0,t)$, we have

$$\begin{aligned} \int_0^t\int_{\mathbb{R}^n}(uu_{tt} + vv_{tt})\mathrm{d}x\mathrm{d}s &= -\int_0^t\int_{\mathbb{R}^n}(|\nabla u|^2 + |\nabla v|^2)\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t\int_{\mathbb{R}^n}[uf_1(u,v) + vf_2(u,v)]\mathrm{d}x\mathrm{d}s \\ &- \int_0^t\int_{\mathbb{R}^n}(|u_t|^{m-2}u_tu + |v_t|^{m-2}v_tv)\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t\int_{\mathbb{R}^n}\int_0^s g(s-\tau)\nabla u(\tau)\cdot\nabla u(s)\mathrm{d}\tau\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t\int_{\mathbb{R}^n}\int_0^s h(s-\tau)\nabla v(\tau)\cdot\nabla v(s)\mathrm{d}\tau\mathrm{d}x\mathrm{d}s. \end{aligned} \tag{18}$$

Thank to Hölder's inequality and Young's inequality, we have

$$\begin{aligned} &\int_0^t\int_{\mathbb{R}^n}\int_0^s g(s-\tau)\nabla u(\tau)\cdot\nabla u(s)\mathrm{d}\tau\mathrm{d}x\mathrm{d}s \\ &\geq \int_0^t\int_0^s g(s-\tau)\|\nabla u(s)\|_2^2\mathrm{d}\tau\mathrm{d}s \\ &\quad - \int_0^t\int_0^s g(s-\tau)\|\nabla u(s)\|_2\|\nabla u(\tau) - \nabla u(s)\|_2\mathrm{d}\tau\mathrm{d}s \\ &\geq \left(1 - \frac{1}{4\zeta}\right)\int_0^t\left(\int_0^s g(\tau)\mathrm{d}\tau\right)\|\nabla u(s)\|_2^2\mathrm{d}s \\ &\quad - \zeta\int_0^t (g\circ\nabla u)(s)\mathrm{d}s \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\int_0^t\int_{\mathbb{R}^n}\int_0^s h(s-\tau)\nabla v(\tau)\cdot\nabla v(s)\mathrm{d}\tau\mathrm{d}x\mathrm{d}s \\ &\geq \left(1 - \frac{1}{4\zeta}\right)\int_0^t\left(\int_0^s h(\tau)\mathrm{d}\tau\right)\|\nabla v(s)\|_2^2\mathrm{d}s \\ &\quad - \zeta\int_0^t (h\circ\nabla v)(s)\mathrm{d}s \end{aligned} \tag{20}$$

for some $\zeta > 0$ to be specified later. By using Young's inequality and (14), we have

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^n}(|u_t|^{m-1}|u|+|v_t|^{r-1}|v|)\mathrm{d}x\mathrm{d}s\le\frac{\varsigma^m}{m}\int_0^t\int_{\mathbb{R}^n}|u|^m\mathrm{d}x\mathrm{d}s+\frac{\delta^r}{r}\int_0^t\int_{\mathbb{R}^n}|v|^r\mathrm{d}x\mathrm{d}s\\&+\max\left\{\frac{m-1}{m}\varsigma^{-m/(m-1)},\frac{r-1}{r}\delta^{-r/(r-1)}\right\}\int_0^t\int_{\mathbb{R}^n}(|u_t|^m+|v_t|^r)\mathrm{d}x\mathrm{d}s\le\frac{\varsigma^m}{m}\int_0^t\int_{\mathbb{R}^n}|u|^m\mathrm{d}x\mathrm{d}s\\&+\frac{\delta^r}{r}\int_0^t\int_{\mathbb{R}^n}|v|^r\mathrm{d}x\mathrm{d}s+\max\left\{\frac{m-1}{m}\varsigma^{-m/(m-1)},\frac{r-1}{r}\delta^{-r/(r-1)}\right\}(J'(t)-J'(0))\end{aligned}\tag{21}$$

for all ς, $\delta>0$. Taking into account (18)–(21) in (17), we obtain

$$\begin{aligned}K'(t)\ge{}&(1-\gamma)J^{-\gamma}(t)J'(t)+\varepsilon\int_{\mathbb{R}^n}(u_0u_1+v_0v_1)\mathrm{d}x+\varepsilon\int_0^t\int_{\mathbb{R}^n}(u_t^2+v_t^2)\mathrm{d}x\mathrm{d}s\\&-\varepsilon\int_0^t\left[1-\left(1-\frac{1}{4\zeta}\right)\int_0^s g(\tau)\mathrm{d}\tau\right]\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x\mathrm{d}s\\&-\varepsilon\int_0^t\left[1-\left(1-\frac{1}{4\zeta}\right)\int_0^s h(\tau)\mathrm{d}\tau\right]\int_{\mathbb{R}^n}|\nabla v|^2\mathrm{d}x\mathrm{d}s+\varepsilon\int_0^t\int_{\mathbb{R}^n}[uf_1(u,v)+vf_2(u,v)]\,\mathrm{d}x\mathrm{d}s\\&-\varepsilon\frac{\varsigma^m}{m}\int_0^t\int_{\mathbb{R}^n}|u|^m\mathrm{d}x\mathrm{d}s-\varepsilon\frac{\delta^r}{r}\int_0^t\int_{\mathbb{R}^n}|v|^r\mathrm{d}x\mathrm{d}s-\varepsilon\max\left\{\frac{m-1}{m}\varsigma^{-m/(m-1)},\frac{r-1}{r}\delta^{-r/(r-1)}\right\}J'(t)\\&+\varepsilon\max\left\{\frac{m-1}{m}\varsigma^{-m/(m-1)},\frac{r-1}{r}\delta^{-r/(r-1)}\right\}J'(0)-\varepsilon\zeta\int_0^t[(g\circ\nabla u)+(h\circ\nabla v)](s)\mathrm{d}s.\end{aligned}\tag{22}$$

By taking ς and δ so that $\varsigma^{-m/(m-1)}=\delta^{-r/(r-1)}=MJ^{-\gamma}(t)$, for large M to be specified later, and substituting in (22) we arrive at

$$\begin{aligned}K'(t)\ge{}&\left[(1-\gamma)-\varepsilon\max\left\{\frac{m-1}{m},\frac{r-1}{r}\right\}M\right]J^{-\gamma}(t)J'(t)+\varepsilon\max\left\{\frac{m-1}{m},\frac{r-1}{r}\right\}MJ^{-\gamma}(t)J'(0)\\&+\varepsilon\int_{\mathbb{R}^n}(u_0u_1+v_0v_1)\mathrm{d}x+\varepsilon\int_0^t\int_{\mathbb{R}^n}(u_t^2+v_t^2)\mathrm{d}x\mathrm{d}s-\varepsilon\int_0^t\left[1-\left(1-\frac{1}{4\zeta}\right)\int_0^s g(\tau)\mathrm{d}\tau\right]\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x\mathrm{d}s\\&-\varepsilon\int_0^t\left[1-\left(1-\frac{1}{4\zeta}\right)\int_0^s h(\tau)\mathrm{d}\tau\right]\int_{\mathbb{R}^n}|\nabla v|^2\mathrm{d}x\mathrm{d}s+\varepsilon\int_0^t\int_{\mathbb{R}^n}[uf_1(u,v)+vf_2(u,v)]\mathrm{d}x\mathrm{d}s\\&-\varepsilon\frac{M^{1-m}}{m}J^{\gamma(m-1)}(t)\int_0^t\int_{\mathbb{R}^n}|u|^m\mathrm{d}x\mathrm{d}s-\varepsilon\frac{M^{1-r}}{r}J^{\gamma(r-1)}(t)\int_0^t\int_{\mathbb{R}^n}|v|^r\mathrm{d}x\mathrm{d}s\\&-\varepsilon\zeta\int_0^t[(g\circ\nabla u)+(h\circ\nabla v)](s)\mathrm{d}s.\end{aligned}\tag{23}$$

From the definition of $J(t)$ and (7), we have

$$J^{\gamma(m-1)}(t)\le2^{\gamma(m-1)-1}\left[\left(\int_0^t\int_{\mathbb{R}^n}I(u,v)\mathrm{d}x\mathrm{d}s\right)^{\gamma(m-1)}+(\rho T+\omega)^{\gamma(m-1)}\left(\int_{\mathbb{R}^n}(u_0^2+v_0^2)\mathrm{d}x\right)^{\gamma(m-1)}\right].$$

Thank to Hölder's inequality and the finite speed propagation, we have

$$\int_0^t\int_{\mathbb{R}^n}|u|^m\mathrm{d}x\mathrm{d}s\le C\int_0^t(R+T)^{n(p-m)/p}\left(\int_{\mathbb{R}^n}|u|^p\mathrm{d}x\right)^{m/p}\mathrm{d}s\le\frac{C(R+T)^{n(p-m)/p}T^{(p-m)/p}}{C_0^{m/p}}\left(\int_0^t\int_{\mathbb{R}^n}I(u,v)\mathrm{d}x\mathrm{d}s\right)^{m/p}.$$

Therefore, we get

$$
\begin{aligned}
&J^{\gamma(m-1)}(t)\int_0^t\int_{\mathbb{R}^n}|u|^m\mathrm{d}x\mathrm{d}s\le\frac{2^{\gamma(m-1)-1}C(R+T)^{n(p-m)/p}T^{(p-m)/p}}{C_0^{m/p}}\\
&\cdot\left[\left(\int_0^t\int_{\mathbb{R}^n}I(u,v)\mathrm{d}x\mathrm{d}s\right)^{\gamma(m-1)+\frac{m}{p}}+(\rho T+\omega)^{\gamma(m-1)}\left(\int_{\mathbb{R}^n}(u_0^2+v_0^2)\mathrm{d}x\right)^{\gamma(m-1)}\left(\int_0^t\int_{\mathbb{R}^n}I(u,v)\mathrm{d}x\mathrm{d}s\right)^{m/p}\right]\\
&\le\frac{2^{\gamma(m-1)-1}C(R+T)^{n(p-m)/p}T^{(p-m)/p}}{C_0^{m/p}}\left[1+(\rho T+\omega)^{\gamma(m-1)}\left(\int_{\mathbb{R}^n}(u_0^2+v_0^2)\mathrm{d}x\right)^{\gamma(m-1)}\right]\left(1+\int_0^t\int_{\mathbb{R}^n}I(u,v)\mathrm{d}x\mathrm{d}s\right)
\end{aligned}
\tag{24}
$$

since $\gamma(m-1)+m/p\le1$. Similarly, we have

$$
\begin{aligned}
J^{\gamma(r-1)}(t)\int_0^t\int_{\mathbb{R}^n}|v|^r\mathrm{d}x\mathrm{d}s&\le\frac{2^{\gamma(r-1)-1}C(R+T)^{n(p-r)/p}T^{(p-r)/p}}{C_0^{r/p}}\\
&\cdot\left[1+(\rho T+\omega)^{\gamma(r-1)}\left(\int_{\mathbb{R}^n}(u_0^2+v_0^2)\mathrm{d}x\right)^{\gamma(r-1)}\right]\left(1+\int_0^t\int_{\mathbb{R}^n}I(u,v)\mathrm{d}x\mathrm{d}s\right).
\end{aligned}
\tag{25}
$$

Inserting (24) and (25) in (23) and choosing

$$
\varepsilon\le\frac{1-\gamma}{\max\left\{\frac{m-1}{m},\frac{r-1}{r}\right\}M},
$$

we find

$$
\begin{aligned}
K'(t)\ge{}&\varepsilon\int_{\mathbb{R}^n}(u_0u_1+v_0v_1)\mathrm{d}x+\varepsilon\int_0^t\int_{\mathbb{R}^n}(u_t^2+v_t^2)\mathrm{d}x\mathrm{d}s-\varepsilon\int_0^t\left[1-\left(1-\frac{1}{4\zeta}\right)\int_0^s g(\tau)\mathrm{d}\tau\right]\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x\mathrm{d}s\\
&-\varepsilon\int_0^t\left[1-\left(1-\frac{1}{4\zeta}\right)\int_0^s h(\tau)\mathrm{d}\tau\right]\int_{\mathbb{R}^n}|\nabla v|^2\mathrm{d}x\mathrm{d}s+\varepsilon pC_1\int_0^t\int_{\mathbb{R}^n}(|u|^p+|v|^p)\mathrm{d}x\mathrm{d}s\\
&-\varepsilon C_1\max\left\{M^{1-m}Q_1(T),M^{1-r}Q_2(T)\right\}\int_0^t\int_{\mathbb{R}^n}(|u|^p+|v|^p)\mathrm{d}x\mathrm{d}s-\varepsilon\max\left\{M^{1-m}Q_1(T),M^{1-r}Q_2(T)\right\}\\
&-\varepsilon\zeta\int_0^t[(g\circ\nabla u)+(h\circ\nabla v)](s)\mathrm{d}s,
\end{aligned}
$$

where

$$
\begin{aligned}
Q_1(T)&=\frac{2^{\gamma(m-1)-1}C(R+T)^{n(p-m)/p}T^{(p-m)/p}}{mC_0^{m/p}}\left[1+(\rho T+\omega)^{\gamma(m-1)}\left(\int_{\mathbb{R}^n}(u_0^2+v_0^2)\mathrm{d}x\right)^{\gamma(m-1)}\right],\\
Q_2(T)&=\frac{2^{\gamma(r-1)-1}C(R+T)^{n(p-r)/p}T^{(p-r)/p}}{rC_0^{r/p}}\left[1+(\rho T+\omega)^{\gamma(r-1)}\left(\int_{\mathbb{R}^n}(u_0^2+v_0^2)\mathrm{d}x\right)^{\gamma(r-1)}\right].
\end{aligned}
$$

From (12) and (7), we may also write

$$
\begin{aligned}
K'(t)\ge{}&\varepsilon\int_{\mathbb{R}^n}(u_0u_1+v_0v_1)\mathrm{d}x+\varepsilon\int_0^t\int_{\mathbb{R}^n}(u_t^2+v_t^2)\mathrm{d}x\mathrm{d}s-\varepsilon\int_0^t\left[1-\left(1-\frac{1}{4\zeta}\right)\int_0^s g(\tau)\mathrm{d}\tau\right]\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x\mathrm{d}s\\
&-\varepsilon\int_0^t\left[1-\left(1-\frac{1}{4\zeta}\right)\int_0^s h(\tau)\mathrm{d}\tau\right]\int_{\mathbb{R}^n}|\nabla v|^2\mathrm{d}x\mathrm{d}s+\varepsilon pC_1\int_0^t\int_{\mathbb{R}^n}(|u|^p+|v|^p)\mathrm{d}x\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&-\varepsilon C_1 \max\left\{M^{1-m}Q_1(T), M^{1-r}Q_2(T)\right\}\int_0^t\int_{\mathbb{R}^n}(|u|^p+|v|^p)\mathrm{d}x\mathrm{d}s-\varepsilon\max\left\{M^{1-m}Q_1(T), M^{1-r}Q_2(T)\right\}\\
&-\varepsilon\zeta\int_0^t[(g\circ\nabla u)+(h\circ\nabla v)](s)\mathrm{d}s+\xi J(t)-\xi\int_0^t\int_{\mathbb{R}^n}I(u,v)\mathrm{d}x\mathrm{d}s+\frac{\xi}{2}\int_0^t\int_{\mathbb{R}^n}(u_t^2+v_t^2)\mathrm{d}x\mathrm{d}s\\
&+\frac{\xi}{2}\int_0^t\left(1-\int_0^s g(\tau)\mathrm{d}\tau\right)\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x\mathrm{d}s+\frac{\xi}{2}\int_0^t\left(1-\int_0^s h(\tau)\mathrm{d}\tau\right)\int_{\mathbb{R}^n}|\nabla v|^2\mathrm{d}x\mathrm{d}s\\
&+\frac{\xi}{2}\int_0^t[(g\circ\nabla u)+(h\circ\nabla v)](s)\mathrm{d}s-\xi(\rho T+\omega)\int_{\mathbb{R}^n}(u_0^2+v_0^2)\mathrm{d}x,
\end{aligned}
$$

where ξ is a positive constant to be determined later. That is

$$
\begin{aligned}
K'(t)\geq{}&\xi J(t)+\left(\frac{\xi}{2}+\varepsilon\right)\int_0^t\int_{\mathbb{R}^n}(u_t^2+v_t^2)\mathrm{d}x\mathrm{d}s\\
&+\varepsilon C_1\left[\left(p-\max\left\{M^{1-m}Q_1(T), M^{1-r}Q_2(T)\right\}\right)-\frac{\xi}{\varepsilon}\right]\int_0^t\int_{\mathbb{R}^n}(|u|^p+|v|^p)\mathrm{d}x\mathrm{d}s\\
&+\varepsilon\int_0^t\left[\frac{\xi}{2\varepsilon}\left(1-\int_0^s g(\tau)\mathrm{d}\tau\right)-\left[1-\left(1-\frac{1}{4\zeta}\right)\int_0^s g(\tau)\mathrm{d}\tau\right]\right]\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x\mathrm{d}s\\
&+\varepsilon\int_0^t\left[\frac{\xi}{2\varepsilon}\left(1-\int_0^s h(\tau)\mathrm{d}\tau\right)-\left[1-\left(1-\frac{1}{4\zeta}\right)\int_0^s h(\tau)\mathrm{d}\tau\right]\right]\int_{\mathbb{R}^n}|\nabla v|^2\mathrm{d}x\mathrm{d}s\\
&+\varepsilon\left(\frac{\xi}{2\varepsilon}-\zeta\right)\int_0^t[(g\circ\nabla u)+(h\circ\nabla v)](s)\mathrm{d}s-\xi(\rho T+\omega)\int_{\mathbb{R}^n}(u_0^2+v_0^2)\mathrm{d}x\\
&+\varepsilon\int_{\mathbb{R}^n}(u_0u_1+v_0v_1)\mathrm{d}x-\varepsilon\max\left\{M^{1-m}Q_1(T), M^{1-r}Q_2(T)\right\}.
\end{aligned}
$$

Choosing $\zeta=\frac{\xi}{2\varepsilon}$ and ξ satisfies $2\varepsilon<\xi<p\varepsilon$, we get that

$$
\begin{aligned}
K'(t)\geq{}&2\varepsilon J(t)+2\varepsilon\int_0^t\int_{\mathbb{R}^n}(u_t^2+v_t^2)\mathrm{d}x\mathrm{d}s\\
&+\varepsilon C_1\left[\left(p-\frac{\xi}{\varepsilon}\right)-\max\left\{M^{1-m}Q_1(T), M^{1-r}Q_2(T)\right\}\right]\int_0^t\int_{\mathbb{R}^n}(|u|^p+|v|^p)\mathrm{d}x\mathrm{d}s\\
&+\varepsilon\int_0^t\left[\left(\frac{\xi}{2\varepsilon}-1\right)-\left(\frac{\xi}{2\varepsilon}-1+\frac{\varepsilon}{2\xi}\right)\left(\int_0^s g(\tau)\mathrm{d}\tau\right)\right]\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x\mathrm{d}s\\
&+\varepsilon\int_0^t\left[\left(\frac{\xi}{2\varepsilon}-1\right)-\left(\frac{\xi}{2\varepsilon}-1+\frac{\varepsilon}{2\xi}\right)\left(\int_0^s h(\tau)\mathrm{d}\tau\right)\right]\int_{\mathbb{R}^n}|\nabla v|^2\mathrm{d}x\mathrm{d}s\\
&+\varepsilon\left[\int_{\mathbb{R}^n}(u_0u_1+v_0v_1)\mathrm{d}x-\xi(\rho T+\omega)\int_{\mathbb{R}^n}(u_0^2+v_0^2)\mathrm{d}x-\max\left\{M^{1-m}Q_1(T), M^{1-r}Q_2(T)\right\}\right].
\end{aligned}
\tag{26}
$$

At this point we choose u_0, u_1, and ξ such that

$$
\int_{\mathbb{R}^n}(u_0u_1+v_0v_1)\mathrm{d}x-\xi(\rho T+\omega)\int_{\mathbb{R}^n}(u_0^2+v_0^2)\mathrm{d}x>0,
\tag{27}
$$

$$
\left(\frac{\xi}{2\varepsilon}-1\right)-\left(\frac{\xi}{2\varepsilon}-1+\frac{\varepsilon}{2\xi}\right)\left(\int_0^s g(\tau)\mathrm{d}\tau\right)>0,\ \text{and}\ \left(\frac{\xi}{2\varepsilon}-1\right)-\left(\frac{\xi}{2\varepsilon}-1+\frac{\varepsilon}{2\xi}\right)\left(\int_0^s h(\tau)\mathrm{d}\tau\right)>0.
$$

This is, of course, possible by (9). Then we pick M large enough so that

$$\left(p-\frac{\xi}{\varepsilon}\right)-\max\left\{M^{1-m}Q_1(T),M^{1-r}Q_2(T)\right\}>0$$

and

$$\int_{\mathbb{R}^n}(u_0u_1+v_0v_1)\mathrm{d}x-\xi(\rho T+\omega)\int_{\mathbb{R}^n}(u_0^2+v_0^2)\mathrm{d}x-\max\left\{M^{1-m}Q_1(T),M^{1-r}Q_2(T)\right\}\geq 0.$$

Therefore (26) takes the form

$$K'(t)\geq\varepsilon\varepsilon'\left\{J(t)+\int_0^t\int_{\mathbb{R}^n}(|u|^p+|v|^p)\mathrm{d}x\mathrm{d}s+\int_0^t\int_{\mathbb{R}^n}(u_t^2+v_t^2)\mathrm{d}x\mathrm{d}s+\int_0^t\int_{\mathbb{R}^n}(|\nabla u|^2+|\nabla v|^2)\mathrm{d}x\mathrm{d}s\right\} \tag{28}$$

for some constant $\varepsilon'>0$. Consequently, we have

$$K(t)>K(0)=\left(\omega\int_{\mathbb{R}^n}(u_0^2+v_0^2)\mathrm{d}x\right)^{1-\gamma}>0\ \text{ for all }\ t\in[0,T].$$

Next we estimate

$$\left|\int_0^t\int_{\mathbb{R}^n}(uu_t+vv_t)\mathrm{d}x\mathrm{d}s\right|\leq\int_0^t\left[\left(\int_{\mathbb{R}^n}u^2\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^n}u_t^2\mathrm{d}x\right)^{1/2}+\left(\int_{\mathbb{R}^n}v^2\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^n}v_t^2\mathrm{d}x\right)^{1/2}\right]\mathrm{d}s$$
$$\leq\ C(R+T)^{n(p-2)/2p}\int_0^t\left[\left(\int_{\mathbb{R}^n}|u|^p\mathrm{d}x\right)^{1/p}\left(\int_{\mathbb{R}^n}u_t^2\mathrm{d}x\right)^{1/2}+\left(\int_{\mathbb{R}^n}|v|^p\mathrm{d}x\right)^{1/p}\left(\int_{\mathbb{R}^n}v_t^2\mathrm{d}x\right)^{1/2}\right]\mathrm{d}s$$

which implies

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{R}^n}(uu_t+vv_t)\mathrm{d}x\mathrm{d}s\right|^{1/(1-\gamma)}&\leq C(R+T)^{\nu}T^{(p-2)/p}\left[\left(\int_0^t\int_{\mathbb{R}^n}|u|^p\mathrm{d}x\mathrm{d}s\right)^{\mu/[p(1-\gamma)]}\right.\\ &\left.+\left(\int_0^t\int_{\mathbb{R}^n}u_t^2\mathrm{d}x\mathrm{d}s\right)^{\theta/[2(1-\gamma)]}+\left(\int_0^t\int_{\mathbb{R}^n}|v|^p\mathrm{d}x\mathrm{d}s\right)^{\mu/[p(1-\gamma)]}+\left(\int_0^t\int_{\mathbb{R}^n}v_t^2\mathrm{d}x\mathrm{d}s\right)^{\theta/[2(1-\gamma)]}\right]\end{aligned} \tag{29}$$

for $1/\mu+1/\theta=1$, where $\nu=n(p-2)/[2p(1-\gamma)]$. We take $\theta=2(1-\gamma)$, to get $\mu/(1-\gamma)=2/(1-2\gamma):=\lambda\leq p$. Therefore, by Lemma 2.7, (29) becomes

$$\begin{aligned}&\left|\int_0^t\int_{\mathbb{R}^n}(uu_t+vv_t)\mathrm{d}x\mathrm{d}s\right|^{1/(1-\gamma)}\leq\\ &C(R+T)^{\nu}T^{(p-2)/p}\left[\int_0^t\int_{\mathbb{R}^n}(u_t^2+v_t^2)\mathrm{d}x\mathrm{d}s+\left(\int_0^t\int_{\mathbb{R}^n}|u|^p\mathrm{d}x\mathrm{d}s\right)^{\lambda/p}+\left(\int_0^t\int_{\mathbb{R}^n}|v|^p\mathrm{d}x\mathrm{d}s\right)^{\lambda/p}\right]\leq\\ &C(R+T)^{\nu}T^{(p-2)/p}\left[\int_0^t\int_{\mathbb{R}^n}(u_t^2+v_t^2)\mathrm{d}x\mathrm{d}s+C'(R+T)^{2/p-2/p^*}\int_0^t\int_{\mathbb{R}^n}(|\nabla u|^2+|\nabla v|^2+|u|^p+|v|^p)\mathrm{d}x\mathrm{d}s\right]\end{aligned} \tag{30}$$

for some constant $C'>0$. Finally, it is clear that

$$K^{1/(1-\gamma)}(t)\leq 2^{1/(1-\gamma)}\left(J(t)+\varepsilon^{1/(1-\gamma)}\left|\int_0^t\int_{\mathbb{R}^n}(uu_t+vv_t)\mathrm{d}x\mathrm{d}s\right|^{1/(1-\gamma)}\right). \tag{31}$$

A combination of (28), (30), and (31) then yields

$$K'(t) \geq \Gamma K^{1/(1-\gamma)}(t), \quad t \leq T,$$

for some constant $\delta > 0$. A direct integration over $(0,t)$ gives

$$K^{\gamma/(1-\gamma)}(t) \geq \frac{1}{K^{-\gamma/(1-\gamma)}(0) - \gamma\Gamma t/(1-\gamma)}, \quad \forall t \leq T. \tag{32}$$

Therefore (32) shows that for ω (introduced in (12)) large enough $K(t)$ blows up in a time

$$t^* \leq \frac{1-\gamma}{\Gamma\gamma\left(\omega\int_{\mathbb{R}^n}(u_0^2+v_0^2)\mathrm{d}x\right)^{\gamma}} \leq T.$$

In fact, we have the above result provide that

$$\omega \geq \left(\frac{1-\gamma}{\Gamma\gamma T}\right)^{\frac{1}{\gamma}}\left(\int_{\mathbb{R}^n}(u_0^2+v_0^2)\mathrm{d}x\right)^{-1}. \tag{33}$$

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