Integrability Test and Travelling-Wave Solutions of Higher-Order Shallow-Water Type Equations

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We apply the Weiss-Tabor-Carnevale (WTC) Painlevé test to members of a sequence of higher-order shallow-water type equations. We obtain the result that the equations considered are non-integrable, although compatibility conditions at real resonances are satisfied. We also construct travelling-wave solutions for these and related equations.

Key words: Integrability Tests; Shallow-Water Equations; Exact Solutions.
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1. Introduction

The derivation of exact solutions of physically-interesting partial differential equations (PDEs) is a topic that has long been of interest, and there are many techniques available in order to realize this aim, for example, the use of Lie symmetries [1 – 3], the variational iteration method [4, 5], and the homotopy perturbation method [6, 7]. Also of great interest, over the last thirty years or so, has been the connection between the integrability of PDEs and analytical properties of their solutions, and in particular the Weiss-Tabor-Carnevale (WTC) Painlevé test [8]. Techniques arising within this context can also be used to derive exact solutions via various so-called truncation procedures [8 – 13].

In the present paper we will be considering the application of the WTC Painlevé test and truncation to the higher-order shallow-water type equations discussed in [14, 15],

$$\begin{aligned} &u_t - u_{xxt} + u_{(2n+1)x} - u_{(2n+3)x} + 3uu_x \\ &\quad - 2u_x u_{xx} - uu_{xxx} = 0, \quad n = 1,2,3,\ldots. \end{aligned} \tag{1}$$

For $n = 0$ this equation is equivalent to the well-known completely integrable Fuchssteiner-Fokas-Camassa-Holm equation [16, 17] to which the WTC Painlevé test has already been carried out in [18], so we exclude this case from further consideration here. Indeed, since those cases having $n \geq 1$ considered below appear to be non-integrable, we find that the nature of the integrable case $n = 0$ differs markedly from that of other members of this sequence.

In Section 2 we apply the WTC Painlevé test to the cases $n = 1,2,\ldots,15$ of equation (1). In Section 3 we derive travelling-wave solutions using the truncation of the WTC Painlevé expansion for the cases $n = 1$ and $n = 2$ and in Section 4 corresponding travelling-wave solutions for related equations. Section 5 is devoted to conclusions.

2. Integrability Test

We now turn to the application of the WTC Painlevé test to the sequence of equations

$$\begin{aligned} K_n[u] \equiv{}& u_t - u_{xxt} + u_{(2n+1)x} - u_{(2n+3)x} + 3uu_x \\ &- 2u_x u_{xx} - uu_{xxx} = 0, \quad n = 1,2,3,\ldots. \end{aligned} \tag{2}$$

This test [8], simplified using Kruskal's 'reduced ansatz' [19], consists of seeking a solution of the form

$$\begin{aligned} &u = \varphi^p \sum_{j=0}^{\infty} u_j \varphi^j, \\ &\text{where } u_j = u_j(t) \text{ and } \varphi = x + \psi(t), \end{aligned} \tag{3}$$

and requires a choice of expansion family or branch, that is, a choice of leading order exponent p, leading order coefficient u_0, and corresponding dominant terms $\hat{K}[u]$. For each family there is a set of indices, or resonances, $\mathfrak{R} = \{r_1,\ldots,r_N\}$, which give the values of j at which arbitrary data are introduced in the expansion (3), or in a suitable modification thereof. For the sequence of equations under consideration (2), it is straight forward to show that the only possible non-trivial expansion family is that having

$$\begin{aligned} &p = -2n, \quad u_0 = -\frac{\prod_{k=2}^{2n+2}(2n+k)}{2(3n+1)}, \\ &\hat{K}[u] = -u_{(2n+3)x} - 2u_x u_{xx} - uu_{xxx}. \end{aligned} \tag{4}$$

We note that this expansion family will have a full complement of $(2n+3)$ resonances. Taking these considerations into account, the single-valuedness requirements of the WTC Painlevé test then are translated into the question if all resonances of the family (4) are distinct integers, and if all corresponding compatibility conditions are satisfied.

The resonance polynomial corresponding to the family (4) can be written as

$$\begin{aligned} P(r;n) &= (r-4n-2) \\ &\cdot \left[\prod_{k=0}^{2n+1} (r-2n-k) - \frac{\prod_{k=2}^{2n+2}(2n+k)}{2(3n+1)} \right. \\ &\left. \cdot \left(r^2-(6n+1)r+4n(3n+1) \right) \right], \end{aligned} \tag{5}$$

and thus we see immediately that $r = 4n+2$ is a resonance, as expected when the dominant terms $\hat{K}[u]$ are a derivative of terms of that weight [20]. Given that $r = -1$ is always a resonance (easily checked), the invariance of the second factor of $P(r;n)$ under $r \to -r+6n+1$ allows us to deduce that $r = 6n+2$ is also a resonance.

We now consider particular choices of n: for the choices considered the aforementioned resonances $r = -1$, $r = 4n+2$, and $r = 6n+2$ are the only real zeros of (5).

2.1. The Case $n = 1$

For $n = 1$ equation (2) reads

$$\begin{aligned} K_1[u] \equiv u_t - u_{xxt} + u_{xxx} - u_{5x} \\ + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0. \end{aligned} \tag{6}$$

The leading-order behaviour and resonance polynomial for (6) are

$$\begin{aligned} &u \sim -15\varphi^{-2}, \\ &P(r,1) = (r+1)(r-6)(r-8)(r^2-7r+15), \end{aligned} \tag{7}$$

and it is straight forward to check that the compatibility conditions at $r = 6$ and $r = 8$ are identically satisfied. The complex roots of the quadratic factor mean that the equation does not pass the WTC Painlevé test and is presumably not integrable.

2.2. The Case $n = 2$

For $n = 2$ equation (2) reads

$$\begin{aligned} K_2[u] \equiv u_t - u_{xxt} + u_{5x} - u_{7x} \\ + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \end{aligned} \tag{8}$$

and the corresponding leading-order behaviour and resonance polynomial are

$$\begin{aligned} &u \sim -2160\varphi^{-4}, \\ &P(r,2) = (r+1)(r-10)(r-14) \\ &\qquad \cdot (r^2-13r+60)(r^2-13r+72). \end{aligned} \tag{9}$$

Once again it is straight forward to check that the compatibility conditions at the real resonances $r = 10$ and $r = 14$ are identically satisfied. The two quadratic factors have complex roots, and the equation thus fails the WTC Painlevé test and is presumably not integrable.

2.3. The Cases $3 \le n \le 15$

For $3 \le n \le 15$ we find that the only real resonances are $r = -1$, $r = 4n+2$, and $r = 6n+2$: the corresponding PDEs fail the WTC Painlevé test and are presumably not integrable, although all compatibility conditions at real resonances are satisfied.

3. Exact Solutions

In this section we seek travelling-wave solutions of the PDE (2) in the special cases $n = 1$ and $n = 2$. We use a truncated WTC expansion in the travelling-wave reduction of (2).

3.1. The Case $n = 1$

We obtain the travelling-wave solution of (6)

$$\begin{aligned} u = \frac{15}{4} k^2 \operatorname{sech}^2 \left(\frac{k}{2}(x-ct-x_0) \right) \\ + \frac{1}{8}(11-10k^2-k^4), \end{aligned} \tag{10}$$

with speed

$$c = \frac{1}{8}(33-k^4), \tag{11}$$

where k is a free parameter.

We note that zero boundary conditions are allowed for real $k = \pm 1$ (the sign is irrelevant), and thus for fixed speed $c = 4$:

$$u = \frac{15}{4} \operatorname{sech}^2 \left(\frac{1}{2}(x-4t-x_0) \right). \tag{12}$$

3.2. The Case n = 2

We obtain the travelling-wave solution of (8)

$$\begin{aligned} u = {} & \frac{5}{6589}(8202 - 8664k^2 + 1083k^4 + 6859k^6) \\ & + \frac{90}{19}k^2(4 + 19k^2)\,\mathrm{sech}^2\left(\frac{k}{2}(x - ct - x_0)\right) \\ & - 135k^4\,\mathrm{sech}^4\left(\frac{k}{2}(x - ct - x_0)\right), \end{aligned} \tag{13}$$

where the speed c is given by

$$c = (110246 - 15162k^4 + 34295k^6)/6859, \tag{14}$$

and k musty satisfy

$$\begin{aligned} & (19k^2 + 4)(20577k^6 + 40793k^4 \\ & \quad -30362k^2 + 6392) = 0. \end{aligned} \tag{15}$$

We note that in this case the wave speed is fixed, and that the solution (13) cannot satisfy zero boundary conditions. We also note that (15) does not admit real solutions, although it does admit purely imaginary solutions. We discuss this further in the next section.

4. Exact Solutions of Related Equations

4.1. The Case n = 1

Here we observe that the solution (10), (11) can also be used to obtain the solution

$$\begin{aligned} u = {} & -\frac{15}{4}\mu^2\,\mathrm{sech}^2\left(\frac{\mu}{2}(\xi - c\tau - \xi_0)\right) \\ & + \frac{1}{8}(11 + 10\mu^2 - \mu^4), \end{aligned} \tag{16}$$

with speed

$$c = \frac{1}{8}(33 - \mu^4), \tag{17}$$

where μ is a free parameter, of the equation

$$\begin{aligned} & u_\tau + u_{\xi\xi\tau} - u_{\xi\xi\xi} - u_{5\xi} \\ & \quad + 3uu_\xi + 2u_\xi u_{\xi\xi} + uu_{\xi\xi\xi} = 0, \end{aligned} \tag{18}$$

by setting

$$x = -\mathrm{i}\xi, \quad t = -\mathrm{i}\tau, \quad x_0 = -\mathrm{i}\xi_0, \quad k = \mathrm{i}\mu. \tag{19}$$

Zero boundary conditions are then allowed for real $\mu = \pm\sqrt{11}$ (again, the sign is irrelevant), and thus for fixed speed $c = -11$:

$$u = -\frac{165}{4}\,\mathrm{sech}^2\left(\frac{\sqrt{11}}{2}(\xi + 11\tau - \xi_0)\right). \tag{20}$$

4.2. The Case n = 2

Using once again the change of variables (19) we find, for the case $n = 2$, the solution

$$\begin{aligned} u = {} & \frac{5}{6589}(8202 + 8664\mu^2 + 1083\mu^4 - 6859\mu^6) \\ & - \frac{90}{19}\mu^2(4 - 19\mu^2)\,\mathrm{sech}^2\left(\frac{\mu}{2}(\xi - c\tau - \xi_0)\right) \\ & - 135\mu^4\,\mathrm{sech}^4\left(\frac{\mu}{2}(\xi - c\tau - \xi_0)\right), \end{aligned} \tag{21}$$

where the speed c is given by

$$c = (110246 - 15162\mu^4 - 34295\mu^6)/6859 \tag{22}$$

and μ musty satisfy

$$\begin{aligned} & (4 - 19\mu^2)(6392 + 30362\mu^2 \\ & \quad +40793\mu^4 - 20577\mu^6) = 0, \end{aligned} \tag{23}$$

of the equation

$$\begin{aligned} & u_\tau + u_{\xi\xi\tau} + u_{5\xi} + u_{7\xi} + 3uu_\xi \\ & \quad + 2u_\xi u_{\xi\xi} + uu_{\xi\xi\xi} = 0. \end{aligned} \tag{24}$$

For this last equation we thus obtain two travelling-wave solutions, given by substituting $\mu = \alpha$ in (21), where α is one of the two positive real roots of (23) (we can choose the positive roots since once again the sign is irrelevant). For example, The choice $\mu = 2/\sqrt{19}$ yields

$$\begin{aligned} u = {} & \frac{4550}{599} \\ & - \frac{2160}{361}\,\mathrm{sech}^4\left[\frac{1}{\sqrt{19}}\left(\xi - \frac{109254}{6859}\tau - \xi_0\right)\right]. \end{aligned} \tag{25}$$

5. Conclusions

We have applied the WTC Painlevé test to a sequence of higher-order shallow-water type equations. Although these equations appear to be non-integrable, we obtain the result that compatibility conditions at real resonances, for the cases $n = 1, 2, \ldots, 15$, are satisfied. We have also used a truncation procedure to obtain travelling-wave solutions in the first two of these cases, and furthermore have seen how complex values of parameters resulting from this process yield solutions of related equations.

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