Numerical Solution of a Boussinesq Type Equation Using Fourier Spectral Methods

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Efficient numerical methods for solving nonlinear wave equations and studying the propagation and stability properties of their solitary waves (solitons) are applied to a Boussinesq type equation in one space dimension. These methods use a pseudospectral (Fourier transform) treatment of the space dependence, together with (a) finite differences, or (b) a fourth-order Runge-Kutta scheme (RK4), for the time evolution. Our schemes follow very accurately single solitons, which are given by simple closed formulas and are known to be stable for all allowed velocities. However, as a parameter of the problem tends to the critical value $b = 0.5$, where the velocity of the exact soliton vanishes, our solutions destabilize due to numerical errors, producing two small solitons in the place of the exact one. On the other hand, when we study the interaction of two such solitons, starting far apart from each other, we find in the b_1, b_2 parameter plane a curve beyond which the solution becomes unstable by exponential blow-up of the amplitudes, independently of our space and time discretization. We claim that this is due to a dynamical resonance rather than the accumulation of numerical errors, in agreement with theoretical predictions. Our implementation relies on the fast Fourier transform (FFT) algorithm and no major differences are observed, when either scheme (a) or (b) is used for the evolution of time.

Key words: Fourier Spectral Method; Fast Fourier Transform; Boussinesq Equation; Nonlinear Waves.

1. Introduction

Discretization using finite differences in time and spectral methods in space has proved to be very useful in solving numerically nonlinear partial differential equations (PDE) describing wave propagation [1, 2]. The Korteweg-de Vries (KdV) equation is one famous example to which such combined schemes have been applied efficiently to analyze unidirectional solitary wave propagation in one dimension [3, 4].

In this paper, we apply a combination of spectral methods and finite differences to another well-known nonlinear PDE of the Boussinesq type. This equation admits bidirectional wave propagation, has closed form solitary wave solutions, and, like the KdV, is completely integrable in one space dimension. These solutions are called solitons and are known to exist in arbitrary number and interact completely elastically. We numerically follow these interactions and investigate their stability properties by varying the velocity parameter of the waves which appears in their analytical form. Our future purpose is to compare these results with those of other numerical schemes, which use e. g. finite differences in both time and space.

Let us consider an equation of the Boussinesq type,

$$u_{tt} - u_{xx} + 3(u^2)_{xx} + u_{xxxx} = 0, \tag{1}$$

with subscripts denoting differentiation with respect to x and t, which was introduced to describe the propagation of long waves in shallow water. Boussinesq type equations arise in several other physical applications including one-dimensional nonlinear lattice waves, vibrations in a nonlinear string, and ion sound waves in a plasma [5, 6].

It is well known that (1) has a bidirectional solitary wave solution

$$u(x,t) = 2b^2 \operatorname{sech}^2(b(x - x_0 - ct)) \tag{2}$$

representing a solitary wave, where $c = \pm\sqrt{1-4b^2}$ is the propagation speed and b, x_0 are arbitrary constants determining the height and the position of the maximum height of the wave, respectively. From the form of c it is apparent that the solution can propagate in either direction (left or right). We should also mention that, in order to have the wave solution (2), parameter b must satisfy the relation $|b| < 0.5$. Moreover, the maximum of the wave $2b^2$ occurs at the point $x = x_0 + ct$. In the present work, we have applied to (1) two numerical methods to study soliton solutions and investigate their interactions upon collision:

(a) a combination of finite differences and a Fourier pseudospectral method and

(b) a combination of classical fourth-order Runge-Kutta scheme (RK4) and a Fourier pseudospectral method.

Both methods were equally successful and yielded very similar results with comparable efficiency and accuracy. In particular, we employed them to check the analytical form of the one-soliton solution (2) and then estimated the region of stability of two such solitons, started far apart from each other and allowed to collide. The boundary we found in the b_1, b_2 parameter plane, beyond which this interaction becomes unstable by amplitude blow-up, coincides with the one published in the recent literature [7] and is also verified theoretically, as this boundary corresponds to a two-soliton resonant interaction discovered a long time ago [8].

2. Fourier Based RK4 Method for Boussinesq Equation

Consider the Boussinesq equation (1) in the following form:

$$u_{tt} = u_{xx} - 6uu_{xx} - 6u_x^2 - u_{xxxx}, \quad x \in [a,z]. \quad (3)$$

We solve this equation first by combining an RK4 method with respect to t and a Fourier pseudospectral method with respect to x. To prepare the equation for numerical solution we introduce the auxiliary variable $v = u_t$. This reduces the second-order equation in time to the first order system

$$u_t = v, \quad v_t = u_{xx} - 6uu_{xx} - 6u_x^2 - u_{xxxx}. \quad (4)$$

The initial conditions we use to solve (4) numerically can be extracted from the above relation (2) for $t = 0$ at $u(x,t)$ and $u_t(x,t)$,

$$\begin{aligned} u(x,0) &= 2b^2 \operatorname{sech}^2(b(x-x_0)), \\ v(x,0) &= u_t(x,0) \\ &= 4b^3 c \operatorname{sech}^2(b(x-x_0)) \tanh(b(x-x_0)). \end{aligned} \quad (5)$$

Next, we change the solution interval from $[a,z]$ to $[0,2\pi]$, using the transformation

$$x \to \frac{2\pi}{L}(x-a),$$

where $L = z - a$ and thus (4) becomes

$$\begin{aligned} u_t &= v, \\ v_t &= \left(\frac{2\pi}{L}\right)^2 u_{xx} - 6\left(\frac{2\pi}{L}\right)^2 uu_{xx} \\ &\quad - 6\left(\frac{2\pi}{L}\right)^2 u_x^2 - \left(\frac{2\pi}{L}\right)^4 u_{xxxx}. \end{aligned} \quad (6)$$

We now transform $u(x,t)$ and derivatives into Fourier space with respect to x, that means applying the inverse Fourier transform

$$u_x = F^{-1}\{\mathrm{i}kF(u)\}, \quad u_{xx} = F^{-1}\{-k^2F(u)\},$$
$$u_{xxxx} = F^{-1}\{k^4F(u)\}.$$

Thus, (6) becomes

$$\begin{aligned} u_t &= v, \\ v_t &= \left(\frac{2\pi}{L}\right)^2 F^{-1}\{-k^2F(u)\} \\ &\quad - 6\left(\frac{2\pi}{L}\right)^2 uF^{-1}\{-k^2F(u)\} \\ &\quad - 6\left(\frac{2\pi}{L}\right)^2 [F^{-1}\{\mathrm{i}kF(u)\}]^2 \\ &\quad - \left(\frac{2\pi}{L}\right)^4 [F^{-1}\{k^4F(u)\}]. \end{aligned} \quad (7)$$

In practice, we need to discretize (7). For any integer $N > 0$, we consider

$$x_j = j\Delta x = \frac{2\pi}{N}, \quad j = 0,1,\cdots,N-1.$$

Let $u(x,t)$ and $v(x,t)$ be the solutions of (6). We first transform them into the discrete Fourier space

$$\begin{aligned} \hat{u}(k,t) &= F(u) \\ &= \frac{1}{N}\sum_{J=0}^{N-1} u(x_j,t)\mathrm{e}^{\mathrm{i}kx_j}, \quad -\frac{N}{2} \le k \le \frac{N}{2} - 1, \end{aligned}$$

$$\begin{aligned} \hat{v}(k,t) &= F(v) \\ &= \frac{1}{N}\sum_{J=0}^{N-1} v(x_j,t)\mathrm{e}^{\mathrm{i}kx_j}, \quad -\frac{N}{2} \le k \le \frac{N}{2} - 1, \end{aligned}$$

hence, using the inversion formula, we get

$$u(x_j,t) = F^{-1}(\hat{u}) = \sum_{k=-N/2}^{N/2-1} \hat{u}(k,t)\mathrm{e}^{\mathrm{i}kx_j}, \quad 0 \le j \le N-1,$$

$$v(x_j,t) = F^{-1}(\hat{v}) = \sum_{k=-N/2}^{N/2-1} \hat{v}(k,t)\mathrm{e}^{\mathrm{i}kx_j}, \quad 0 \le j \le N-1.$$

Replacing F and F^{-1} in (7) by their discrete counterparts, and discretizing (7) gives

$$\begin{aligned}\frac{\mathrm{d}u(x_j,t)}{\mathrm{d}t} &= v(x_j,t),\\ \frac{\mathrm{d}v(x_j,t)}{\mathrm{d}t} &= \left(\frac{2\pi}{L}\right)^2 F^{-1}\{-k^2F(u)\}\\ &\quad -6\left(\frac{2\pi}{L}\right)^2 u(x_j,t)F^{-1}\{-k^2F(u)\}\\ &\quad -6\left(\frac{2\pi}{L}\right)^2 [F^{-1}\{\mathrm{i}kF(u)\}]^2\\ &\quad -\left(\frac{2\pi}{L}\right)^4 [F^{-1}\{k^4F(u)\}].\end{aligned} \tag{8}$$

By letting

$$\mathbf{U} = [u(x_0,t), u(x_1,t), \cdots, u(x_{N-1},t)]^T$$
$$\mathbf{V} = [v(x_0,t), v(x_1,t), \cdots, v(x_{N-1},t)]^T$$
$$\mathbf{w} = \begin{bmatrix}\mathbf{U}\\ \mathbf{V}\end{bmatrix}, \quad \mathbf{w_t} = \begin{bmatrix}\mathbf{U}_t\\ \mathbf{V}_t\end{bmatrix},$$

the system of (8) can be written in the vector form

$$\mathbf{w}_t = \mathbf{F}(\mathbf{U},\mathbf{V}), \tag{9}$$

where $\mathbf{F}$ defines the right hand side of (8).

3. Combination of Finite Differences and a Fourier Pseudospectral Method

The first numerical scheme we shall use is based on a combination of finite differences and a Fourier pseudospectral method. After we have changed the solution interval from $[a,z]$ to $[0,2\pi]$ and transformed $u(x,t)$ and its derivatives into Fourier space with respect to x, we then discretize (3) as follows:

$$\begin{aligned}\frac{\mathrm{d}^2u(x_j,t)}{\mathrm{d}t^2} &= \left(\frac{2\pi}{L}\right)^2 F^{-1}\{-k^2F(u)\}\\ &\quad -6\left(\frac{2\pi}{L}\right)^2 u(x_j,t)F^{-1}\{-k^2F(u)\}\\ &\quad -6\left(\frac{2\pi}{L}\right)^2 [F^{-1}\{\mathrm{i}kF(u)\}]^2\\ &\quad -\left(\frac{2\pi}{L}\right)^4 [F^{-1}\{k^4F(u)\}].\end{aligned} \tag{10}$$

Next, by letting $\mathbf{U} = [u(x_0,t), u(x_1,t), \cdots, u(x_{N-1},t)]^T$, (10) can be written in the vector form

$$\mathbf{U}_{tt} = \mathbf{F}(\mathbf{U}), \tag{11}$$

where $\mathbf{F}$ defines the right hand side of (10). The time derivative in (11) is discretized using a finite difference [9] approximation, in terms of central differences

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{(\Delta t)^2} = \mathbf{F}(\mathbf{u})$$

or

$$\begin{aligned}&u(x,t+\Delta t) = 2u(x,t) - u(x,t-\Delta t)\\ &+ (\Delta t)^2\Bigg[\left(\frac{2\pi}{L}\right)^2 F^{-1}\{-k^2F(u)\}\\ &-6\left(\frac{2\pi}{L}\right)^2 u(x_j,t)F^{-1}\{-k^2F(u)\}\\ &-6\left(\frac{2\pi}{L}\right)^2 [F^{-1}\{\mathrm{i}kF(u)\}]^2 - \left(\frac{2\pi}{L}\right)^4 [F^{-1}\{k^4F(u)\}]\Bigg].\end{aligned} \tag{12}$$

We now need two level initial conditions $u(x,-\Delta t)$ and $u(x,0)$. Using $u(x,0)$, as given in (5), from the central difference of $u_t(x,t)$, where Δt is the time step, we obtain the approximation

$$u^{n-1} = u^{n+1} - 2\Delta t u_t(x,t)$$

from which we get $u(x,-\Delta t)$ to

$$\begin{aligned}u(x,-\Delta t) &= u(x,\Delta t) - 2\Delta t u_t(x,0)\\ &= u(x,\Delta t) + 2\Delta t[4b^3c\,\mathrm{sech}^2(b(x-x_1))\\ &\quad \cdot \tanh(b(x-x_1))].\end{aligned}$$

Substitute $u(x,-\Delta t)$ and $u(x,0)$ in (12) to get $u(x,\Delta t)$:

$$\begin{aligned} u(x,\Delta t) &= u(x,0) + \Delta t u_t(x,0) \\ &+ \frac{(\Delta t)^2}{2}\left[\left(\frac{2\pi}{L}\right)^2 F^{-1}\{-k^2F(u)\}\right. \\ &- 6\left(\frac{2\pi}{L}\right)^2 u(x_j,0)F^{-1}\{-k^2F(u)\} \\ &- 6\left(\frac{2\pi}{L}\right)^2 [F^{-1}\{\mathrm{i}kF(u)\}]^2 \\ &\left. - \left(\frac{2\pi}{L}\right)^4 [F^{-1}\{k^4F(u)\}]\right]. \end{aligned} \quad (13)$$

Finally, we substitute $u(x,0)$ and $u(x,\Delta t)$ in (12) to get $u(x,2\Delta t)$ and so on, until we get $u(x,t)$ at time t. In our studies, we have used various values of N (128 to 1024) and time steps, ranging from $\Delta t = 0.0001$ to 0.02.

4. Numerical Results and Examples

In order to show how good the numerical solutions are in comparison with the exact ones, we will use L_2 and L_∞ error norms.

$$L_2 = \|u^{\text{exact}} - u^{\text{num}}\|_2 = \left[\Delta x \sum_{i=1}^{N} |u_i^{\text{exact}} - u_i^{\text{num}}|^2\right]^{1/2},$$

$$L_\infty = \|u^{\text{exact}} - u^{\text{num}}\|_\infty = \max_i |u_i^{\text{exact}} - u_i^{\text{num}}|,$$

see Table 1.

To evaluate the performance of our methods, three problems have been considered: the motion of a single

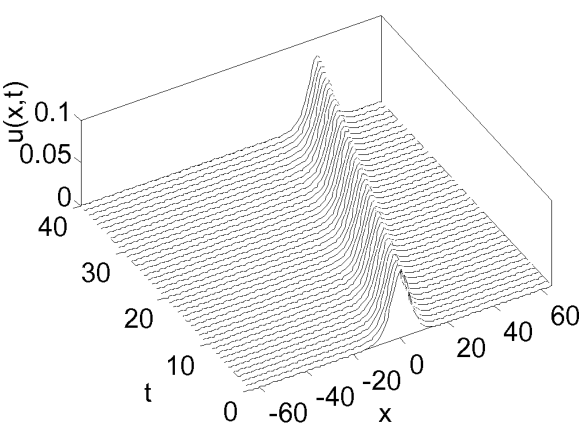

Fig. 1. Numerical simulation of the Fourier spectral solution of the Boussinesq equation using RK4 scheme with $b = 0.2$, $c = +\sqrt{1-4b^2}$, and $N = 128$.

Table 1. Error norms of the Fourier spectral solution of the Boussinesq equation using RK4 scheme: (a) $t = 40$ with $N =$ 128, 256, and 512; (b) $t = 10$, 20, 30, and 40 with $N = 128$.

(a)

N	$L_2(\times 10^{-3})$	$L_\infty(\times 10^{-3})$	Amplitude
128	12.6338	4.3381	0.0792055
256	6.3224	2.1867	0.0795634
512	3.1624	1.0945	0.0797424

(b)

t	$L_2(\times 10^{-3})$	$L_\infty(\times 10^{-3})$
10	3.3286	1.2964
20	6.4411	2.3372
30	9.5471	3.3493
40	12.6338	4.3381

Table 2. Error norms of the Fourier spectral solution of the Boussinesq equation using a finite difference scheme: (a) $t =$ 40 with $N = 128$, 256, 512, and 1024; (b) $t = 10$, 20, 30, and 40 with $N = 128$.

(a)

N	$L_2(\times 10^{-3})$	$L_\infty(\times 10^{-3})$	Amplitude
128	12.6671	4.3895	0.0792055
256	6.3558	2.2006	0.0795644
512	3.1957	1.1056	0.0797412
1024	1.6148	0.5583	0.0798291

(b)

t	$L_2(\times 10^{-3})$	$L_\infty(\times 10^{-3})$
10	3.3500	1.3185
20	6.4407	2.3477
30	9.5468	3.3693
40	12.6671	4.3895

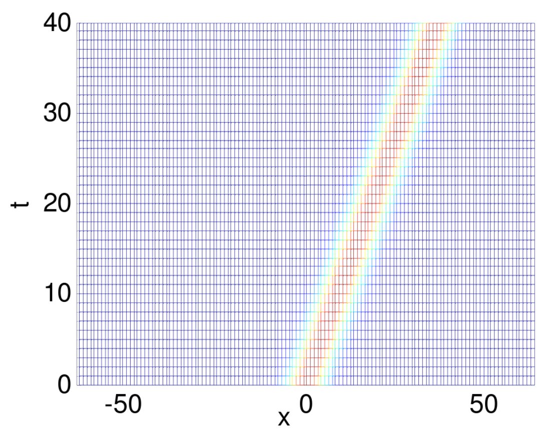

Fig. 2. Plan view of the Fourier spectral solution of the Boussinesq equation using RK4 scheme with $b = 0.2$, $c = +\sqrt{1-4b^2}$, and $N = 128$.

solitary wave in right direction, the motion of a single solitary wave in left direction, and the collision of two solitary waves moving in opposite directions. Several

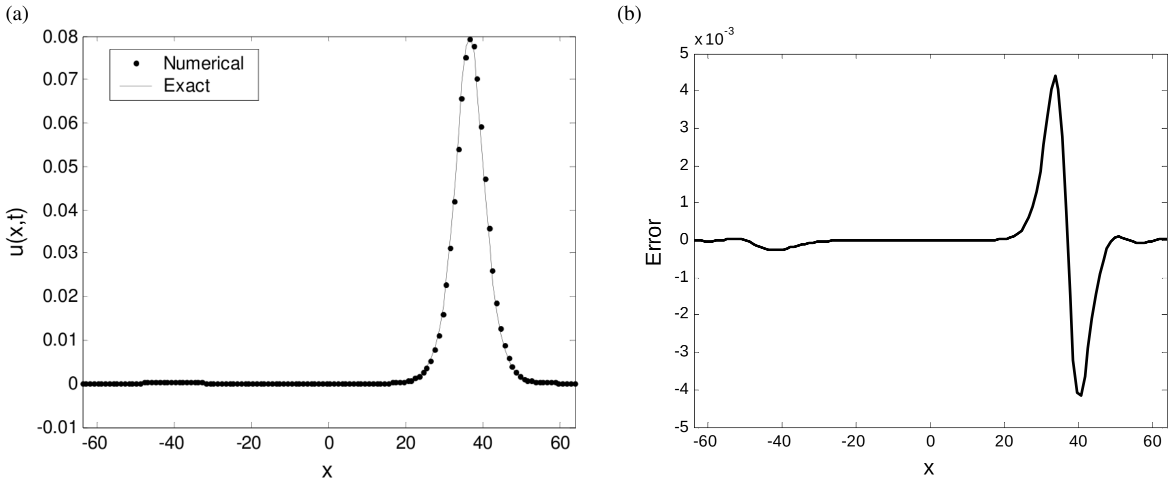

Fig. 3. (a) Fourier spectral solution and (b) error distributions in the Fourier spectral solution of the Boussinesq equation using RK4 scheme at $t = 40$ with $b = 0.2$, $c = +\sqrt{1-4b^2}$, and $N = 128$.

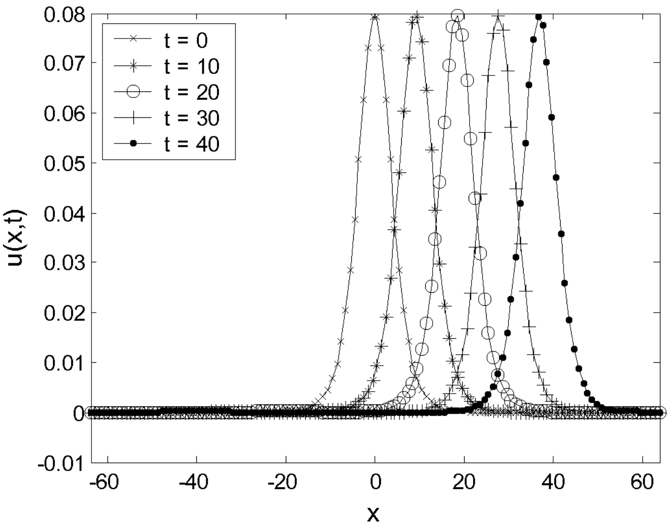

Fig. 4. Fourier spectral solution of the Boussinesq equation using a finite difference scheme at $t = 0$, 10, 20, 30, and 40 with $b = 0.2$, $c = +\sqrt{1-4b^2}$, and $N = 128$.

tests were made first to compute the single soliton solution (2) of the Boussinesq equation, using a combination of finite differences and the Fourier pseudospectral method, which verified that the above errors were essentially of the same order of magnitude for various values of N (128 to 1024) and time steps $\Delta t = 0.0001$ to 0.02. In Figures 1 – 6 we have plotted these solutions and checked them against the exact formula (2) for the one-soliton solution of the Boussinesq equation (1), moving either to the right or the left. Very similar results were obtained when the Fourier pseudospectral method was combined with an RK4 scheme in time (see Table 2).

As b increases, however, numerical errors accumulate and for values of $|b|$ close to 0.5, our solution is seen to deviate from the exact formula, producing two smaller solitons in the place of a single one, as shown in Figure 7. We conclude, therefore, for a tolerable accuracy of our solutions, that the maximum value of b we should allow in our experiments is 0.4.

4.1. The Motion of a Single Solitary Wave

We now solve explicitly the Boussinesq equation (3) using the periodic boundary conditions

$$u(a,t) = u(z,t) = 0$$

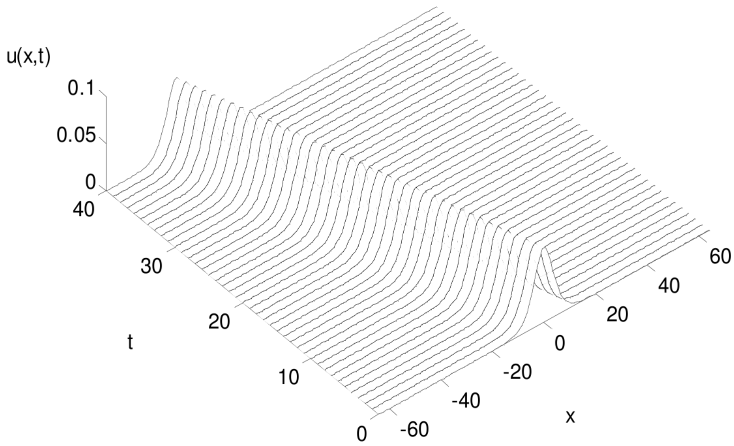

Fig. 5. Numerical simulation of the Fourier spectral solution of the Boussinesq equation using RK4 scheme with $c = -\sqrt{1-4b^2}$ and $N = 128$.

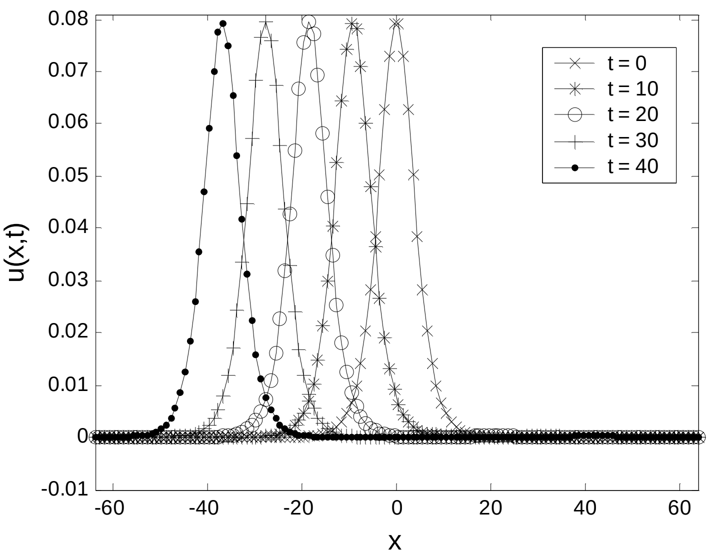

Fig. 6. Fourier spectral solution of the Boussinesq equation using a finite difference scheme at $c = -\sqrt{1-4b^2}$, $t = 0$, 10, 20, 30, and 40 with $N = 128$.

and the initial conditions (5) with $b = 0.2$, $x_1 = 0$, $c = +\sqrt{1-4b^2}$, and $\Delta x = 1$ with $\Delta t = 0.001$. As is clear from Figures 1 – 4, the proposed method follows the propagation of a solitary wave satisfactorily, as it moves to the right with the same amplitude and velocity. The results are the same when the Fourier spectral solution of (3) is combined either with an RK4 scheme or finite differences.

We now solve the same problem for a soliton moving to the left with $b = 0.2$, $x_1 = 0$, $c = -\sqrt{1-4b^2}$, and $\Delta x = 1$ with $\Delta t = 0.001$. It is clear from Figures 5 and 6 that the proposed method again reproduces satisfactorily the evolution of a solitary wave, preserving its shape and velocity, while again the results are the same, when Fourier spectral methods are used with an RK4 scheme or finite differences.

4.2. The Interaction of Two Solitary Waves

The interaction of two positive solitary waves is studied by using the initial condition given by the linear sum of two separate solitary waves of various amplitudes:

$$\begin{aligned} u(x,0) &= u_1(x,0) + u_2(x,0), \\ u(x,0) &= 2{b_1}^2 \operatorname{sech}^2(b_1(x-x_1)) \\ &\quad + 2{b_2}^2 \operatorname{sech}^2(b_2(x-x_1)), \qquad (14) \\ u_t(x,0) &= 4{b_1}^3 c_1 \operatorname{sech}^2(b_1(x-x_1)) \tanh(b_1(x-x_1)) \\ &\quad + 4{b_2}^3 c_2 \operatorname{sech}^2(b_2(x-x_2)) \tanh(b_2(x-x_2)). \end{aligned}$$

We solve (3) with $b_1 = 0.15$, $x_1 = 40$, $c_1 = +\sqrt{1-4{b_1}^2}$, $b_2 = 0.1$, $x_2 = 180$, $c_2 = -\sqrt{1-4{b_2}^2}$,

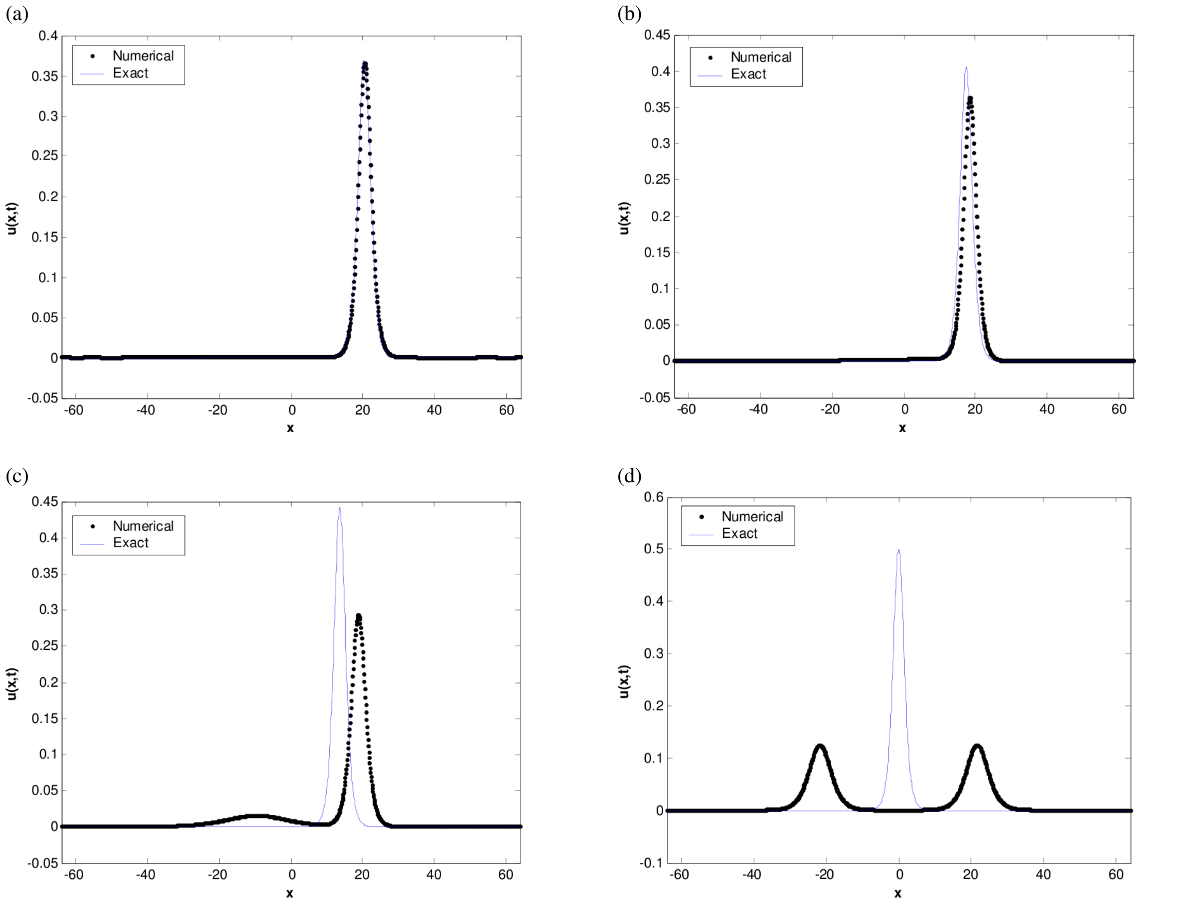

Fig. 7. Fourier spectral solution of the Boussinesq equation using finite difference scheme at $t = 40$ with $c = +\sqrt{1-4b^2}$, $N = 1024$, $\Delta x = 0.125$, $\Delta t = 0.001$, $x_1 = 0$, and (a) $b = 0.43$, (b) $b = 0.45$, (c) $b = 0.47$, (d) $b = 0.5$.

and $\Delta x = 1$, using the RK4 scheme to propagate the solution in time, together with a Fourier spectral method in space. Figures 8 and 9 show a three-dimensional plan view of the solution $u(x,t)$ in x, t, and u space, using $N = 256$. Finally, in Figure 10, we plot the numerical solutions in x and u space, at $t = 0$, 60, 70, 85, 95, and 120, with $\Delta t = 0.001$ and $N = 256$. The first solitary wave was placed initially on the left side of the region, with smaller amplitude than the second, which was placed on the right, while both waves moved in opposite directions. The shapes of the two solitary waves are graphed during the interaction and after the interaction, which is seen to have separated the larger wave from the smaller one, while they both emerge from the interaction, resuming their former shapes and amplitudes.

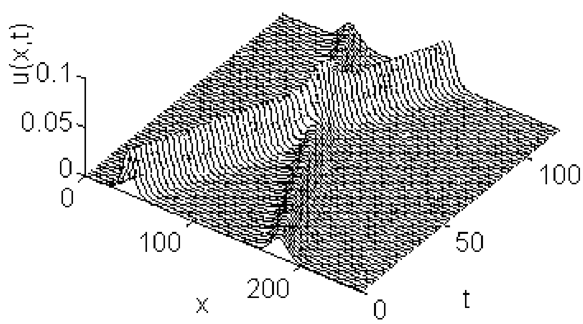

Fig. 8. Numerical simulation of the Fourier spectral solution of a two wave interaction of the Boussinesq equation using RK4 scheme with $N = 256$.

Now, as the two parameters, b_1 and b_2, increase towards the critical value 0.5, where the velocity of the soliton becomes 0, our simulations showed that a very interesting blow-up phenomenon occurs, which may not be an artifact of numerical errors, but a true instability of these solutions, as other authors have

Table 3. Stability region for two wave interaction of the Boussinesq equation.

b_1	b_2	stable / unstable (+/−)
0.05	0.30 or less	+
0.05	0.35	+
0.05	0.40	+
0.05	0.41 or more	−
0.10	0.35 or less	+
0.10	0.36	+
0.10	0.37	+
0.10	0.38 or more	−
0.15	0.30 or less	+
0.15	0.33	+
0.15	0.34 or less	+
0.15	0.35 or more	−
0.20	0.29	+
0.20	0.30 or more	−

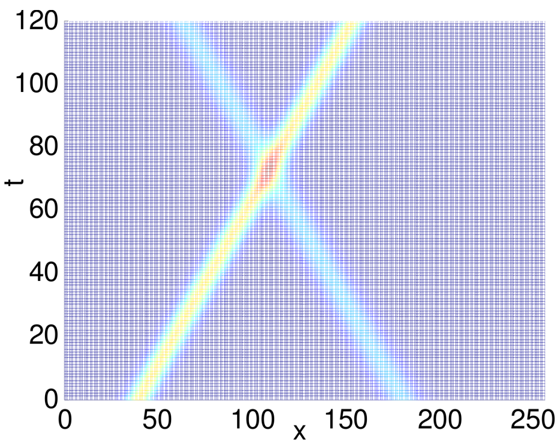

Fig. 9. Plan view of the Fourier spectral solution of a two wave interaction of the Boussinesq equation using RK4 scheme with $N = 256$.

found [7, 8]. Indeed, when we fix one of these parameters, say b_1, and increase b_2, we reach a value of b_2, where, during the interaction of the solitary waves, the corresponding solutions blow up, with their amplitudes reaching very high values. This phenomenon occurs in much the same way, regardless of N or the size of the time step Δt we use. In Table 3 we list some of these critical parameter values where blow up occurs.

Since, our computations and error analysis of the single soliton solutions showed that, near criticality, numerical errors do not grow exponentially, we conjecture that what we have found is a true instability of the system, occurring along a smooth curve in the b_1, b_2 plane, which we plot in Figure 11. This result verifies a similar finding reported in the literature, where Fourier pseudospectral methods were also employed [7].

More importantly, however, our computations appear to agree very well with a theoretical prediction by Tajiri and Nishitani [8], who have predicted that a two-soliton interaction of our Boussinesq equation (3) can become resonant, when the b_1, b_2 and velocity parameters c_1, c_2 satisfy the condition

$$(c_1 - c_2)^2 - 12(b_1 + b_2)^2 = 0. \qquad (15)$$

Indeed, when we insert in (15) the expressions of c_1, c_2 in terms of b_1, b_2, i. e. $c_1 = +\sqrt{1 - 4b_1{}^2}$ and $c_2 = -\sqrt{1 - 4b_2{}^2}$, we obtain a curve in the b_1, b_2 plane, which is very well approximated by the one we have plotted in Figure 11. According to Tajiri and Nishitani, the exact two-soliton solution of (3) blows up exponentially along this curve, exactly as we have discovered with our numerical methods.

5. Conclusions

In this paper, we have used spectral collocation (or pseudospectral) methods, combined with various temporal discretization techniques to numerically compute solitary wave solutions of a Boussinesq type equation. More specifically, we have implemented a fast Fourier transform to discretize the space variable and finite differences or a Runge-Kutta scheme of fourth order (RK4) to follow the time evolution. Both schemes were equally successful and yielded similar results, indicating that they are stable, for a wide range of time steps and number of Fourier exponentials used in the spatial decomposition.

First, we implemented them to reproduce the analytical form of the single soliton solution travelling to the right and left and found that they were able to do so very well, until a parameter b of the system comes close to the critical value $b = 0.5$, beyond which the soliton ceases to exist. In particular, for $b \geq 0.43$, our errors give rise to a second small wave, "bifurcating", to the left and leading ultimately to two identical solitary waves propagating symmetrically away from the true soliton.

We then studied the evolution of two such solitons, started far apart from each other and allowed to collide. Varying their b_1, b_2 parameters between 0 and 0.5, we found a curve in the b_1, b_2 parameter plane, beyond which this interaction becomes unstable by amplitude blow-up. This instability boundary, in fact, coincides with one computed by other researchers, but is also verified theoretically, as it corresponds to a two-soliton

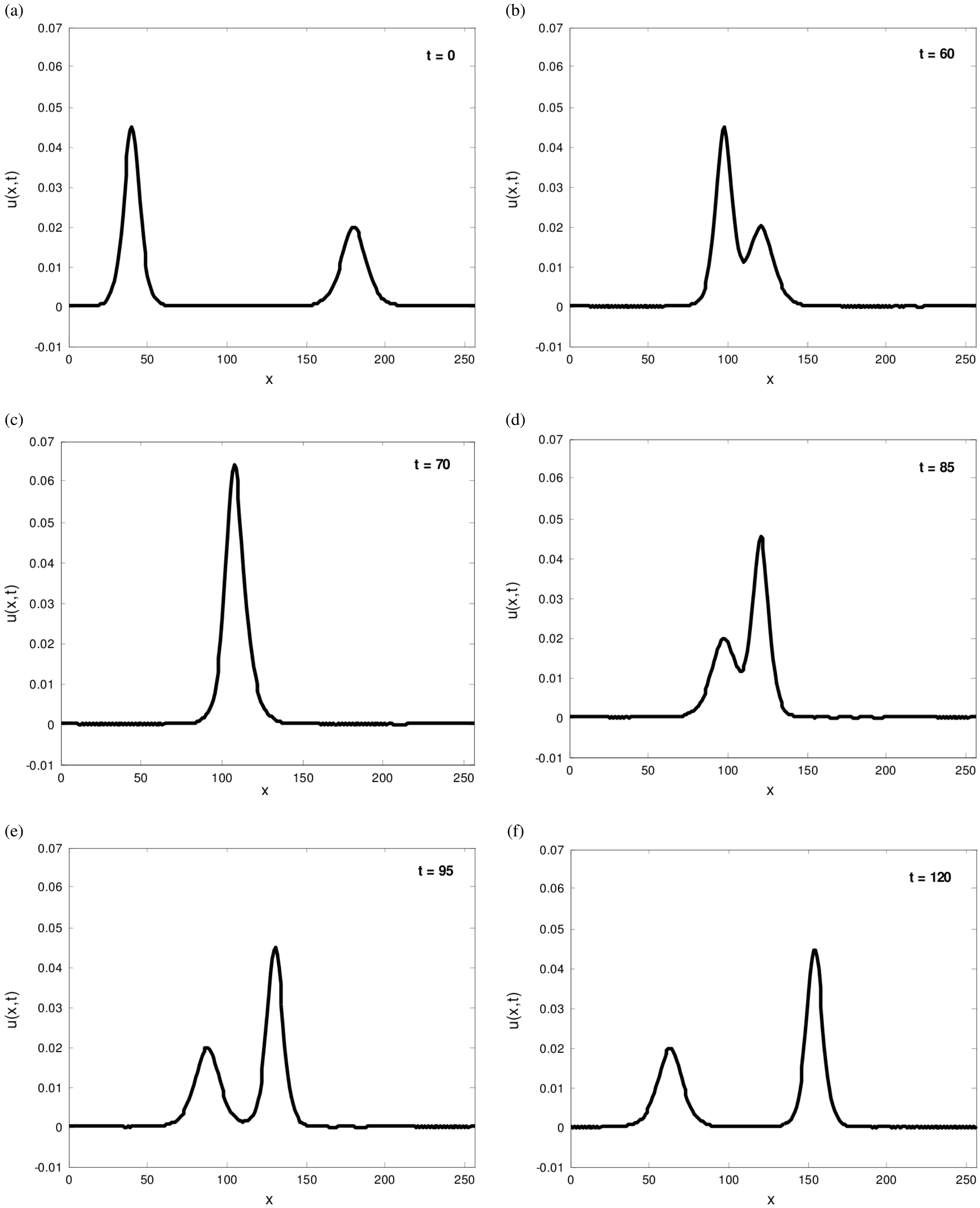

Fig. 10 Fourier spectral solution of a two-wave interaction of the Boussinesq equation using RK4 scheme with $N = 256$.

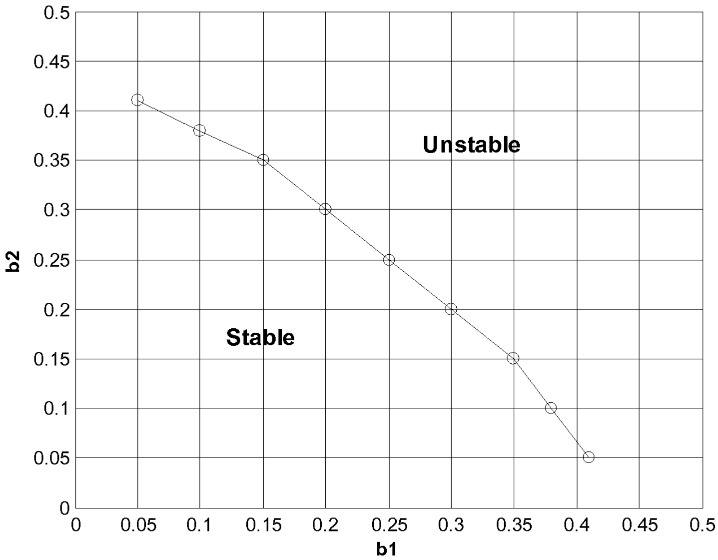

Fig. 11. Stability region for two-soliton interaction of the Boussinesq equation.

resonant interaction known in the literature, since a long time ago. We, thus, conclude that Fourier spatial decomposition methods, combined with different time discretizations, offer a very useful tool for solving numerically nonlinear wave equations of the Boussinesq type, which are bidirectional, as they involve only second-order derivatives with respect to time.

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[1] B. Fornberg, A Practical Guide to Pseudospectral Methods, Cambridge University Press, New York 1996.
[2] J. P. Boyd, Chebyshev and Fourier Spectral Methods, Second Edition Dover Publications Inc., New York 2000.
[3] B, Fornberg and G. B. Whitham, Philos. Trans. Soc. London, **289**, 373 (1978).
[4] E. Tzirtzilakis, V. Marinakis, C. Apokis, and T. Bountis, J. Math. Phys. **43**, 6151 (2002).
[5] G. B. Whitham, Linear and Nonlinear Waves, John Wiley & Sons, 1974.
[6] Z. Yan, Comput. Phys. Commun. **149**, 11 (2002).
[7] E. Tzirtzilakis, C. Skokos, and T. Bountis, Numerical Solution of the Boussinesq equation using Spectral Methods and Stability of Solitary Wave Propagation, Intern. Conf. Proc. From Scientific Computing to Computational Engineering, Athens, 8 – 10 September, 2004.
[8] M. Tajiri and T. Nishitani, J. Phys. Soc. Jpn. **51**, 3720 (1982).
[9] R. J. LeVeque, Finite Difference Methods for Ordinary and Partial Differential Equations, SIAM, Philadelphia, 2007. See also: R. J. LeVeque, Finite Difference Methods for Differential Equations, A Math 585 – 6 Notes, Winter Quarter 2006, University of Washington, Version of January, 2006.