Approximate Solution of Generalized Ginzburg-Landau-Higgs System via Homotopy Perturbation Method

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Using the homotopy perturbation method, a class of nonlinear generalized Ginzburg-Landau-Higgs systems (GGLH) is considered. Firstly, by introducing a homotopic transformation, the nonlinear problem is changed into a system of linear equations. Secondly, by selecting a suitable initial approximation, the approximate solution with arbitrary degree accuracy to the generalized Ginzburg-Landau-Higgs system is derived. Finally, another type of homotopic transformation to the generalized Ginzburg-Landau-Higgs system reported in previous literature is briefly discussed.

Key words: Homotopy Perturbation Method; GLH System; Approximate Solution.
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1. Introduction

As is well known many dynamical problems in physics and other natural fields are usually characterized by nonlinear evolution partial differential equations called as governing equations. In modern nonlinearity the study of exact or approximate solutions and related issues of the construction of solutions to a wide class of nonlinear physical systems have become one of the most exciting and extremely active areas of current investigation [1 – 3]. Recently, many generalized nonlinear systems with practically physical backgrounds were derived and have attracted much more attention both for physicists and mathematicians [4 – 6]. For instance, the nonlinear Ginzburg-Landau-Higgs (GLH) equation with imaginary mass in the real scaler case [7],

$$\phi_{tt} - \phi_{xx} + \phi^3 - \phi = 0, \tag{1}$$

was once derived to describe the drift cyclotron waves for a coherent ion-cyclotron wave in a radially inhomogeneous plasma [7]. In (1), ϕ is the electrostatic potential of the ion-cyclotron wave, x and t are normalized space and time coordinates, respectively. Simple sech and tanh type solitary wave solutions of the GLH equation were obtained and shown to be unstable against small perturbations [7, 8]. The soliton solution of the GLH equation was also derived by using the Hirota bilinear method [9]. The analytical solutions of the GLH equation have been generated by different methods [10], and numerical solutions with particular initial conditions have been implemented by the decomposition method [11].

In this paper, using a perturbed theory, we consider the approximate solution of a generalized Ginzburg-Landau-Higgs (GGLH) equation [12], i. e.,

$$\phi_{tt} - r\phi_{xx} - p\phi + q\phi^3 = f(x,t,\phi,\phi_x,\phi_t), \tag{2}$$

where p, q, r are positive model parameters, $f(x,t,\phi,\phi_x,\phi_t)$ is the system perturbed term and is an analytical function of indicated arguments. If the perturbed term $f(x,t,\phi,\phi_x,\phi_t) = 0$ and $p = q = r = 1$, then the GGLH equation will be reduced to a typical Ginzburg-Landau-Higgs equation (1).

2. Homotopy Perturbation Method and Approximate Solution of the GGLH Equation

In this section, we will give an approximate solution to the GGLH equation via homotopy perturbation method (HPM), which was proposed by He in 2000 [13]. Recently, other expressions like homotopic mapping or homotopic solving method or Mo's perturbation theory appear in Mo's publications [6] and [14]. Actually, the so-called homotopic mapping or homotopic solving method or Mo's Perturbation theory is exactly the homotopy perturbation method, which is still in a in-depth development [15–18].

Here we introduce a homotopic transformation first [19,20], $H(\phi,s):\mathbb{R}\times I\to\mathbb{R}$, namely,

$$\begin{aligned}H(\phi,s)&=\mathcal{L}(\phi)-\mathcal{L}(\varphi)\\&\quad+s[\mathcal{L}(\varphi)+q\phi^3-f(x,t,\phi,\phi_x,\phi_t)],\end{aligned}\tag{3}$$

where $\mathbb{R}=(-\infty,\infty)$, $I=[0,1]$, and s is a homotopic factor ($s\in[0,1]$). The linear operator $\mathcal{L}$ is defined as

$$\mathcal{L}(\phi)=\phi_{tt}-r\phi_{xx}-p\phi.\tag{4}$$

φ is an initial approximate solution of the GGLH equation and satisfies the equation $\mathcal{L}(\varphi)=-q\varphi^3$, which is a typical nonlinear Ginzburg-Landau-Higgs equation. One of the travelling solitary wave solutions of the GLH equation reads [21]

$$\begin{aligned}&\varphi(x,t)=\varepsilon\sqrt{\frac{p}{q}}\tanh\sqrt{\frac{p}{2(1-v^2)}}\left(\sqrt{r}(x-x_0)+vt\right),\\&v^2<1,\ \varepsilon^2=1,\end{aligned}\tag{5}$$

where x_0 and v are the initial wave position and the travelling wave speed, respectively.

Comparing the homotopic mapping (3) with the GGLH Equation (2), we readily find that (2) can follow from (3) when taking $H(\phi,1)=0$. So, the solution of the GGLH Equation (2) is equivalent to the solution of $H(\phi,s)=0$ when $s\to1$. We assume an ansatz ϕ in the following Taylor series form:

$$\phi(x,t,s)=\sum_{j=0}^{\infty}\phi_j(x,t)s^j,\tag{6}$$

where $\phi_j(x,t)=\frac{1}{j!}\frac{\partial^j\phi}{\partial s^j}|_{s=0}$.

Now substituting the ansatz (6) into (3) and collecting the coefficients of polynomials of s, then setting each coefficient to zero yields a system of linear equations,

$$\begin{aligned}&\mathcal{L}(\phi_0)-\mathcal{L}(\varphi)=0,\\&\mathcal{L}(\phi_1)+\mathcal{L}(\varphi)+q\phi_0^3-f(x,t,\phi_0,\phi_{0x},\phi_{0t})=0,\\&\quad\vdots\\&\mathcal{L}(\phi_n)-\psi_n(x,t)-\chi_n(x,t)=0,\quad n=2,3,\cdots,\end{aligned}\tag{7}$$

where

$$\psi_n(x,t)=\frac{-q}{(n-1)!}\frac{\partial^{n-1}}{\partial s^{n-1}}\left[\sum_{j=0}^{\infty}\phi_j(x,t)s^j\right]^3\Bigg|_{s=0},\tag{8}$$

$$\begin{aligned}\chi_n(x,t)=\frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial s^{n-1}}f\Bigg[x,t,\sum_{j=0}^{\infty}\phi_js^j,\\\left(\sum_{j=0}^{\infty}\phi_js^j\right)_x,\left(\sum_{j=0}^{\infty}\phi_js^j\right)_t\Bigg]\Bigg|_{s=0}.\end{aligned}\tag{9}$$

In terms of (7), we can obtain the zeroth-order approximate solution to the GGLH equation, i.e.,

$$\begin{aligned}\phi_0(x,t)&=\varphi\\&=\varepsilon\sqrt{\frac{p}{q}}\tanh\sqrt{\frac{p}{2(1-v^2)}}\left(\sqrt{r}(x-x_0)+vt\right),\\&\quad v^2<1,\ \varepsilon^2=1.\end{aligned}\tag{10}$$

Considering $\mathcal{L}(\varphi)=-q\varphi^3$ and $\varphi=\phi_0$, the first-order approximate solution becomes

$$\mathcal{L}(\phi_1)-f(x,t,\phi_0,\phi_{0x},\phi_{0t})=0,\tag{11}$$

which is a linear hyperbolic type equation and its solution can be derived via Fourier transform approach,

$$\begin{aligned}\phi_1(x,t)=\frac{1}{4\pi}\int_0^t\mathrm{d}\tau\int_{-\infty}^{\infty}F_1(x-\xi,\tau)\\\cdot[G(\xi,t-\tau)-G(\xi,-(t-\tau))]\mathrm{d}\xi,\end{aligned}\tag{12}$$

where $F_1(x-\xi,\tau)=f(x-\xi,\tau,\phi_0,\phi_{0\xi},\phi_{0\tau})$ and

$$G(\xi,t-\tau)=\int_{-\infty}^{\infty}\frac{\exp\left[\sqrt{p-r\lambda^2}(t-\tau)+\mathrm{i}\xi\lambda\right]}{\sqrt{p-r\lambda^2}}\mathrm{d}\lambda,\tag{13}$$

with the initial conditions $\phi_1(x,t)|_{t=0}=0$ and $\phi_{1t}(x,t)|_{t=0}=0$.

In a similar way, we can obtain $\phi_n(x,t),(n\ge2)$, for

$$\begin{aligned}\phi_n(x,t)=\frac{1}{4\pi}\int_0^t\mathrm{d}\tau\int_{-\infty}^{\infty}F_n(x-\xi,\tau)\\\cdot[G(\xi,t-\tau)-G(\xi,-(t-\tau))]\mathrm{d}\xi,\end{aligned}\tag{14}$$

where $G(\xi,t-\tau)$ is given by (13) and $F_n(x-\xi,\tau)=\varphi_n(x-\xi,\tau)+\psi_n(x-\xi,\tau)$, φ_n and ψ_n are determined by (8) and (9), respectively. For example, as $n=2$, $\psi_2(x,t)=3k^3\phi_0^2\phi_1$, and $\chi_2(x,t)=-g_\phi(\phi_0,\varepsilon)\phi_1$, then

$F_2(x,t) = 3k^3\phi_0^2\phi_1 - g_\phi(\phi_0,\varepsilon)\phi_1$, where ϕ_0 and ϕ_1 have been presented by (10) and (12), respectively. According to (14), $\phi_2(x,t)$ can be readily derived. Similarly we can obtain $\phi_3(x,t)$, $\phi_4(x,t)$, $\cdots$.

Finally substituting (10), (12), and (14) into (6) and taking $s \to 1$, the corresponding approximate solution of the GGLH system reads

$$\begin{aligned}\phi(x,t) &= \lim_{s\to 1}\phi(x,t,s) = \sum_{j=0}^{\infty}\phi_j(x,t)\\ &= \varepsilon\sqrt{\frac{p}{q}}\tanh\sqrt{\frac{p}{2(1-v^2)}}\left(\sqrt{r}(x-x_0)+vt\right)\\ &\quad+\frac{1}{4\pi}\sum_{j=1}^{\infty}\int_0^t \mathrm{d}\tau\int_{-\infty}^{\infty}F_n(x-\xi,\tau)\\ &\quad\cdot[G(\xi,t-\tau)-G(\xi,-(t-\tau))]\mathrm{d}\xi,\end{aligned} \tag{15}$$

$v^2 < 1,\ \varepsilon^2 = 1.$

3. Summary and Conclusion

In summary, with the aid of the homotopy perturbation method, an approximate solution with arbitrary order of the GGLH equation is derived. If the perturbed term f in (2) is given concretely, for example $f = \varepsilon\phi^3$, then we can obtain a corresponding approximate solution in terms of formula (18). The homotopy perturbation method is a simple and powerful method, which can be applied in other nonlinear systems. The accuracy of the approximate solution to the original model via the homotopy perturbation method is determined by the initial solution ϕ_0 and the order n. In the paper, we take the initial solution ϕ_0 as a solution of the typical nonlinear Klein-Gordon (NKG) equation $\mathcal{L}(\phi_0) = -q\phi_0^3$, from which we can quickly obtain the approximate solution with arbitrary order to the GGLH equation. The homotopy perturbation method is an approximate analytic method differing from the general numerical methods, which means that the expansion of the approximate solution derived from the homotopy perturbation method can be performed to analytic analysis further, such as qualitative and/or quantitative investigations.

In [14], the authors consider the following GGLH equation:

$$\phi_{tt} - \phi_{xx} + k^2\phi^3 - m^2\phi = g(\phi,\varepsilon), \tag{16}$$

where m, k, ε are model parameters, $g(u,\varepsilon)$ is the system perturbed term and an analytical function of indicated arguments. Now the authors proposed another type of homotopic mapping for the above GGLH equation (16), $H(\phi,p): \mathbb{R}\times I \to \mathbb{R}$,

$$H(\phi,p) = \mathcal{L}(\phi) + k^2\varphi^3 - \mathcal{N}_0(\varphi) + p[\mathcal{N}_0(\varphi) - g(\phi,\varepsilon)], \tag{17}$$

where $\mathbb{R} = (-\infty,\infty)$ and $I = [0,1]$. The linear operator $\mathcal{L}$ and the nonlinear operator $\mathcal{N}_0$ are defined as

$$\mathcal{L}(\phi) = \phi_{tt} - \phi_{xx} - m^2\phi, \tag{18}$$

$$\mathcal{N}_t(\varphi) = \varphi_{tt} - \varphi_{xx} - m^2\varphi + k^2\varphi^3, \tag{19}$$

where φ is an initial approximate solution of the generalized nonlinear Klein-Gordon (GNKG) equation, i. e., $\varphi = \phi_0$. In this way, one can obtain another type of partial differential equations based on the homotopic mapping (17) as $H(\phi,1) = 0$,

$$\phi_{tt} - \phi_{xx} - m^2\phi + k^2\phi_0^3 = g(\phi,\varepsilon), \tag{20}$$

which is not the original GNKG equation. Therefore the derived approximate solution reported in [14] is not a correct solution to the original GNKG equation (16).

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