Analytic Solution of the Sharma-Tasso-Olver Equation by Homotopy Analysis Method

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An analytic technique, the homotopy analysis method (HAM), is applied to obtain the kink solution of the Sharma-Tasso-Olver equation. The homotopy analysis method is one of the analytic methods and provides us with a new way to obtain series solutions of such problems. HAM contains the auxiliary parameter $\hbar$ which gives us a simple way to adjust and control the convergence region of series solution. "Due to this reason, it seems reasonable to rename $\hbar$ the convergence-control parameter" [1].

Key words: Homotopy Analysis Method; Sharma-Tasso-Olver Equation; Kink Solution.

1. Introduction

It is difficult to solve nonlinear problems, especially by analytic techniques. The homotopy analysis method is employed for analytic solutions. The method was introduced first by Liao in 1992 [2, 3]. You can see the application of it in various nonlinear problems in science and engineering, such as the magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet [4] and finding the solutions of the generalized Benjamin-Bona-Mahony equation [5]. All of these successful applications verified the validity, effectiveness, and flexibility of the HAM and recent publication in this topic underline this [6 – 19]. This paper is concerned with the Sharma-Tasso-Olver (STO) equation

$$u_t + \alpha(u^3)_x + \frac{3}{2}\alpha(u^2)_{xx} + \alpha u_{xxx} = 0, \tag{1}$$

where α is a real parameter and $u(x,t)$ is the unknown function depending on the temporal variable t and the spatial variable x. Attention has been focused on STO equation (1) in [20 – 24] and references therein due to its scientific applications. In [20], Yan investigated (1) by using the Cole-Hopf transformation method. In [24], this equation was handled by using the tanh method and the extended tanh method. However, in [21 – 24] the simple symmetry reduction procedure, the Hirota direct method, Bäcklund transformation method, and the extended hyperbolic function method are applied to this equation, respectively. In this paper, we employ the homotopy analysis method to obtain the solitary wave solutions of (1) with unknown wave speed. In the next section, the method will be reviewed briefly.

2. Mathematical Formulation and Solution by Homotopy Analysis Method (HAM)

Under the transformation $\xi = x - pt$ and $u(x,t) = w(\xi)$, (1) reads

$$-pw' + \alpha(w^3)' + \frac{3}{2}\alpha(w^2)'' + \alpha w''' = 0. \tag{2}$$

Integrating (2) once and taking the integration constant equal to zero yields

$$-pw + \alpha w^3 + 3\alpha w w' + \alpha w'' = 0, \tag{3}$$

where p is the velocity of the travelling wave and the prime denotes differentiation with respect to ξ. The boundary conditions for the equation are

$$w(0) = 0, \quad w(\infty) = 1. \tag{4}$$

According to (3) and the boundary conditions (4), the solitary solution can be expressed by

$$w(\xi) = \sum_{m=0}^{+\infty} d_m e^{-m\xi}, \tag{5}$$

where d_m ($m = 0,1,\ldots$) are coefficients to be determined. According to the rule of solution expression denoted by (5) and the boundary conditions (4), it is natural to choose $w_0(\xi) = 1 - \mathrm{e}^{-\xi}$ as initial approximation of $w(\xi)$. Also, under the rule of solution expression (5), it is obvious to choose the auxiliary linear operator

$$L[\phi(\xi;q)] = \left(\frac{\partial^2}{\partial\xi^2} - 1\right)\phi(\xi;q) \tag{6}$$

with the property

$$L[c_1\mathrm{e}^{\xi} + c_2\mathrm{e}^{-\xi}] = 0, \tag{7}$$

where c_1, c_2 are constants. From (3) we define a nonlinear operator

$$N[\phi(\xi;q),P(q)] = -P(q)\phi + \alpha\phi^3 + 3\alpha\phi\frac{\partial\phi}{\partial\xi} + \alpha\frac{\partial^2\phi}{\partial\xi^2} \tag{8}$$

and then construct a homotopy

$$H[\phi(\xi;q),P(q)] = (1-q)L[\phi(\xi;q) - w_0] - \hbar q\mathcal{H}(\xi)N[\phi(\xi;q),P(q)],$$

where $\mathcal{H}(\xi)$ is an auxiliary function. Setting $H[\phi(\xi;q),P(q)] = 0$, we have the zero-order deformation equation

$$(1-q)L[\phi(\xi;q)-w_0] = q\hbar\mathcal{H}(\xi)N[\phi(\xi;q),P(q)], \tag{9}$$

subject to the boundary conditions

$$\phi(0;q) = 0, \quad \phi(\infty;q) = 1, \tag{10}$$

where $\hbar$ is a non-zero auxiliary parameter and $q \in [0,1]$ is the homotopy parameter [1]. When the parameter q increases from 0 to 1, the homotopy solution $\phi(\xi;q)$ varies from $w_0(\xi)$ to $w(\xi)$, so does $P(q)$ from p_0, the initial guess of the wave speed, to p. If this continuous variation is smooth enough, the Maclurin's series with respect to q can be constructed for $\phi(\xi;q)$ and $P(q)$, and further, if these two series are convergent at $q = 1$, we have

$$w(\xi) = w_0(\xi) + \sum_{m=1}^{+\infty} w_m(\xi), \quad p = p_0 + \sum_{m=1}^{+\infty} p_m, \tag{11}$$

where

$$w_m(\xi) = \frac{1}{m!}\frac{\partial^m\phi(\xi;q)}{\partial q^m}\Big|_{q=0}, \qquad p_m = \frac{1}{m!}\frac{\partial^m P(q)}{\partial q^m}\Big|_{q=0}. \tag{12}$$

In [1], $w_m(\xi)$ and p_m are called mth-order homotopy derivatives of $w(\xi)$ and p, respectively. Briefly speaking, by means of the HAM, one constructs a continuous mapping of an initial guess approximation to the exact solution of considered equations [1]. Differentiating (9) and the boundary conditions (10) m times with respect to q, then setting $q = 0$, and finally dividing them by $m!$, we gain the mth-order deformation equation

$$L[w_m(\xi) - \chi_m w_{m-1}(\xi)] = \hbar\mathcal{H}(\xi)R_m(\xi), \tag{13}$$

$$w_m(0) = 0, \quad w_m(\infty) = 0, \tag{14}$$

where

$$R_m(\xi) = \sum_{n=0}^{m-1}\left(-p_{m'}w_n + \alpha w_{m'}\sum_{i=0}^{n} w_i w_{n-i} + 3\alpha w_{m'}w'_n\right) + \alpha w''_{m-1}, \tag{15}$$

with $m' = m - n - 1$ and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m \ge 2. \end{cases}$$

The general solution of (13) is

$$w_m(\xi) = \hat{w}_m(\xi) + c_1\mathrm{e}^{\xi} + c_2\mathrm{e}^{-\xi}, \tag{16}$$

where c_1, c_2 are constants and $\hat{w}_m(\xi)$ is a special solution of (13) which contains the unknown terms p_{m-1}. They are known by solving $w_m(\xi)$, except for p_{m-1}. Under the rule of solution expression and by choosing $\mathcal{H}(\xi) = \mathrm{e}^{\xi}$, we can determine p_{m-1} by vanishing the coefficient of e^{ξ} in $R_m(\xi)$ in each iteration. When $m = 1$, this algebraic equation is

$$\alpha - p_0 = 0.$$

As mentioned above, the general solution of (13) is (16). The unknown c_1, according to the rule of solution expression, is zero and c_2, according to boundary conditions (14), is governed by

$$c_2 = -\hat{w}_m(0),$$

in each iteration.

3. Numerical Results

Liao [3] proved that as long as a series solution given by the HAM converges, it must be one of the exact solutions. So, it is important to ensure that the

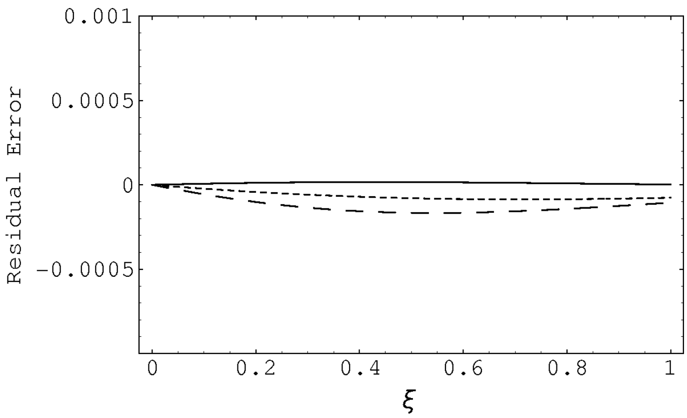

Fig. 1. Curve of the approximation of wave speed p versus $\hbar$ for the 20th-order approximation.

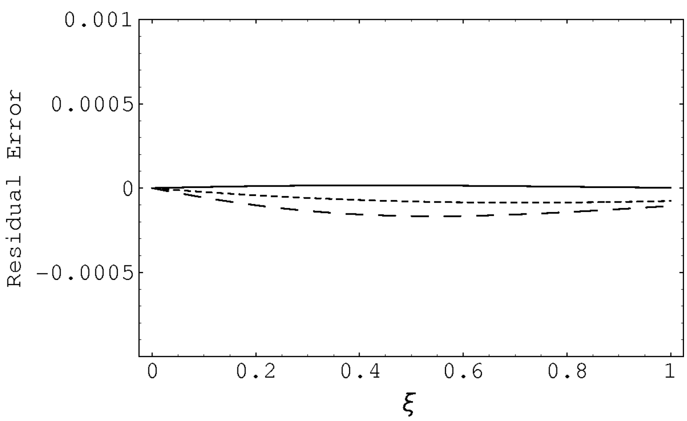

Fig. 2. Residual error of (3) for the 20th-order approximation. Solid curve: $\hbar = -1.4$; dotted curve: $\hbar = -1$; dashed curve: $\hbar = -1.2$.

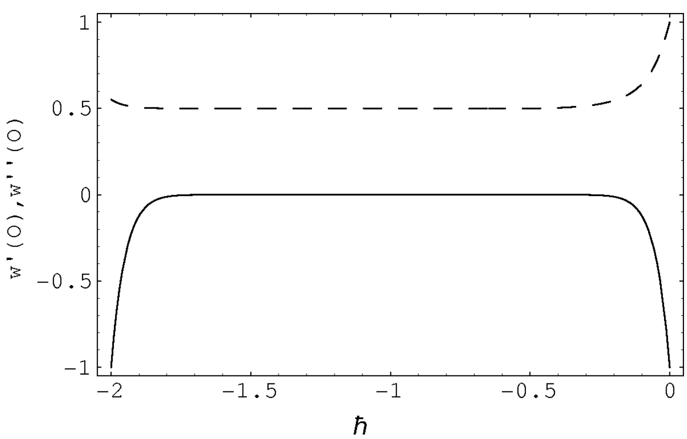

Fig. 3. Curves of approximation of $w'(0)$ and $w''(0)$ versus $\hbar$ for the 20th-order approximation. Solid curve: $w''(0)$; dashed curve: $w'(0)$.

solution series (11) are convergent. We use the widely applied symbolic computation software Mathematica to solve the first few equations of (16). At the Mth-order approximation, we have the approximate analytic solution of (9), namely

$$\begin{aligned} w(\xi) \approx W_M(\xi) &= \sum_{m=0}^{M} w_m(\xi), \\ p \approx \quad P_M \quad &= \sum_{m=0}^{M} p_m. \end{aligned} \tag{17}$$

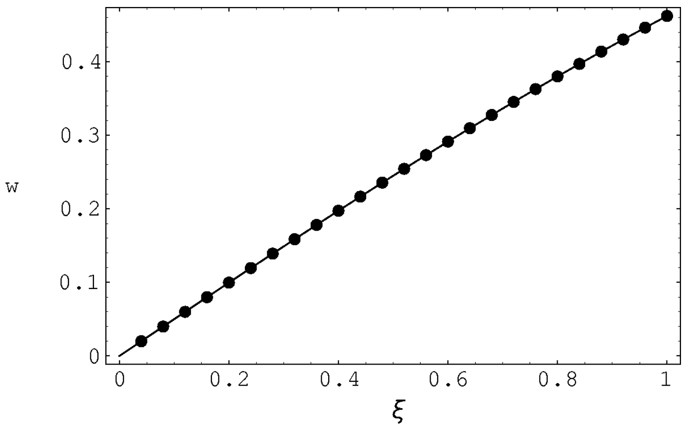

Fig. 4. Analytic approximation for $w(\xi)$ when $\hbar = -1.2$ and the kink solution $w(\xi) = \tanh(\xi/2)$. Solid curve: kink solution; Symbols: 20-th order approximation.

For $\alpha = 1$ we show a few terms of the series solution:

$$w_1(\xi) = \frac{\hbar}{3}\mathrm{e}^{-\xi} - \frac{1}{3}\hbar\mathrm{e}^{-2\xi}$$

$$w_2(\xi) = (\frac{\hbar}{3} + \frac{\hbar^2}{8})\mathrm{e}^{-\xi} - \frac{1}{3}\hbar\mathrm{e}^{-2\xi} - \frac{\hbar^2}{8}\mathrm{e}^{-3\xi}$$

$$w_3(\xi) = \left(\frac{\hbar}{3} + \frac{\hbar^2}{4} + \frac{83\hbar^3}{1440}\right)\mathrm{e}^{-\xi} + \left(\frac{-\hbar}{3} + \frac{\hbar^3}{24}\right)\mathrm{e}^{-2\xi} + \left(\frac{-\hbar^2}{4} - \frac{5\hbar^3}{96}\right)\mathrm{e}^{-3\xi} - \frac{17\hbar^3}{360}\mathrm{e}^{-4\xi}$$

$$\vdots$$

Our solution series contain the auxiliary parameter $\hbar$. As mentioned before the auxiliary parameter $\hbar$ can be employed to adjust the convergence region of the series (11) in the homotopy analysis solution. By means of the so-called $\hbar$-curve, it is straightforward to choose an appropriate range for $\hbar$ which ensures the convergence of the solution series. As pointed out by Liao [3], the appropriate region for $\hbar$ is a horizontal line segment. We can investigate the influence of $\hbar$ on the convergence of p by plotting the curves of p versus $\hbar$, as shown in Figure 1. Generally, it is found that as long as the series solution for the wave speed p is convergent, the corresponding series solution for $w(\xi)$ is also convergent [3]. For instance, our analytic solution converges, as shown by the residual error in Figure 2. It shows the curve for different $\hbar$, which shows the efficiency of HAM. The residual error is defined as follows:

$$\text{Error} \approx \alpha W_M'' + 3\alpha W_M W_M' + \alpha W_M^3 - P_M W_M.$$

In the first three figures we set $\alpha = 1$ for convenience. We can investigate the influence of $\hbar$ on the convergence of $w'(0)$ and $w''(0)$ by plotting the curve of them versus $\hbar$, as shown in Figure 3. The series $w'(0)$ and $w''(0)$ given by the solution series (17) are convergent when $-1.5 \le \hbar \le -0.5$. The series solution for the wave speed is convergent, and the corresponding series for $w(\xi)$ is also convergent. For instance when $\hbar = -1.2$ our analytic solution converges, as shown in Figure 4.

Now we want to investigate the effect of the parameter α on the solutions. By setting different values for α we achieve different bounds for $\hbar$. For example set $\alpha = 3$, once again we can investigate the influence of $\hbar$ on the convergence of p by plotting the curves of p versus $\hbar$, as shown in Figure 5. Consequently, it is found that as long as the series solution for the wave speed p is convergent, the corresponding series solution for $w(\xi)$ is also convergent. For instance, our analytic solution converges, as shown by the residual error in Figure 6. Further, from Figure 7, it is clear that the series $w'(0)$ and $w''(0)$ given by the solution series (17) are convergent when $-0.6 \le \hbar \le -0.1$.

4. Conclusion

In this work, the homotopy analysis method (HAM) [3] is applied to obtain the solution of the Sharma-Tasso-Olver equation. HAM provides us with a convenient way to control the convergence of approximation series by adapting $\hbar$, which is a fundamental qualitative difference in analysis between HAM and other methods. So, this paper, again, shows the flexibility and potential of the homotopy analysis method for nonlinear problems in science and engineering.

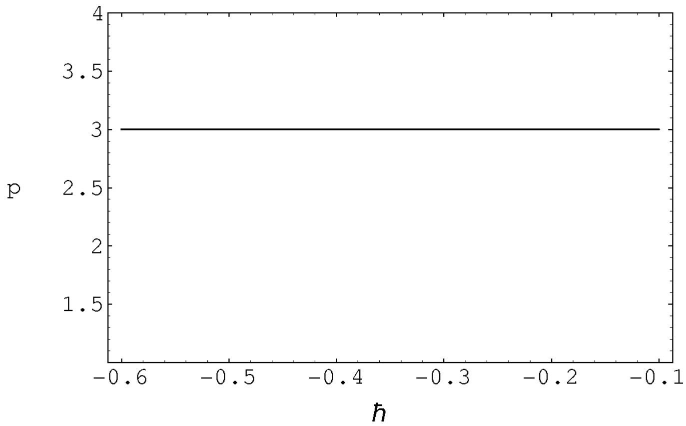

Fig. 5. Curve of the approximation of wave speed p versus $\hbar$ for the 20th-order approximation.

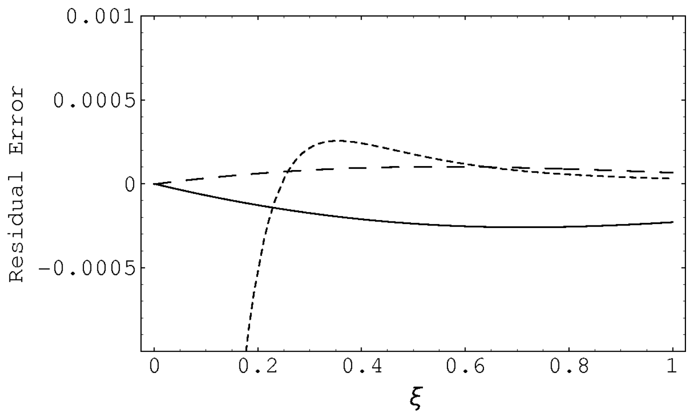

Fig. 6. Residual error of (3) for the 20th-order approximation. Solid curve: $\hbar = -0.4$; dotted curve: $\hbar = -0.5$; dashed curve: $\hbar = -0.6$.

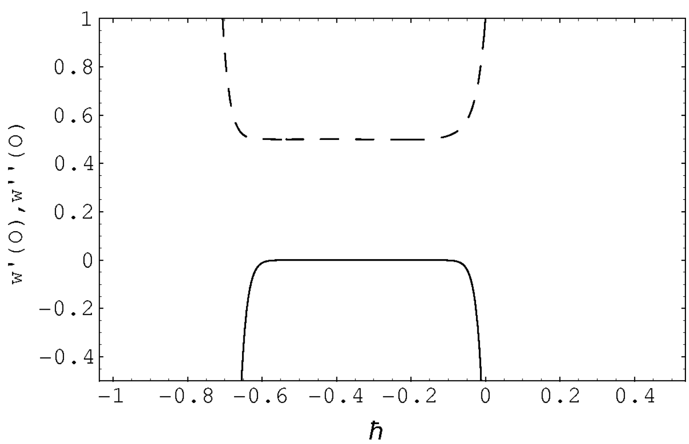

Fig. 7. Curves of approximation of $w'(0)$ and $w''(0)$ versus $\hbar$ for the 20th-order approximation. Solid curve : $w''(0)$; dashed curve: $w'(0)$.

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