Modified Variational Iteration Method for Integro-Differential Equations and Coupled Systems

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In this paper, we apply the modified variational iteration method (mVIM) for solving integro-differential equations and coupled systems of integro-differential equations. The proposed modification is made by the elegant coupling of He's polynomials and the correction functional of variational iteration method. The proposed mVIM is applied without any discretization, transformation or restrictive assumptions and is free from round off errors and calculation of the so-called Adomian's polynomials.

Key words: Variational Iteration Method; He's Polynomials; Integro Differential-Equations; Blasius Problem; Error Estimates.
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1. Introduction

The integro-differential equations are of great significance and are the governing equations of diversified nonlinear physical problems related to physics, astrophysics, magnetic dynamics, water surface, gravity waves, ion acoustic waves in plasma, electromagnetic radiation reactions, engineering, and applied sciences [1 – 35]. Several techniques including decomposition, homotopy perturbation, polynomial and non-polynomial spline, Sink Galerkin, perturbation, homotopy analysis, finite difference, and modified variational iteration have been employed to solve such problems, see [6, 15, 16, 18, 23, 24, 35] and the reference therein. Most of these used schemes are coupled with inbuilt deficiencies like calculation of the so-called Adomian's polynomials and non-compatibility with the physical nature of the problems. He [16 – 32] developed the variational iteration and homotopy perturbation methods which have been applied [1 – 35] to a wide class of nonlinear problems. In a later work Ghorbani et. al. [12, 13] introduced He's polynomials which are calculate from He's homotopy perturbation method. Recently, Noor and Mohyud-Din [30, 32 – 34] made the elegant coupling of He's polynomials and the correction functional of variational iteration method and called it as modified variational iteration method (mVIM). It has to be highlighted that the modified version can also be applied to fractional differential equations and inverse problems [9, 10]. The basic motivation of the present work is the implementation of the modified variational iteration method (mVIM) for solving integro-differential equations and coupled systems of integro-differential equations. The numerical results are very encouraging.

2. Variational Iteration Method (VIM)

To illustrate the basic concept of He's VIM, we consider the following general differential equation:

$$Lu + Nu = g(x), \tag{1}$$

where L is a linear operator, N a nonlinear operator, and $g(x)$ is the inhomogeneous term. According to variational iteration method [1 – 11, 14, 16, 18, 23 – 28, 30, 31 – 35], we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \left(Lu_n(s) + N\tilde{u}_n(s) - g(s)\right)\mathrm{d}s, \tag{2}$$

where λ is a Lagrange multiplier [16, 18, 23 – 28], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation, i. e. $\delta\tilde{u}_n = 0$; (2) is called a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the

Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in [13–15, 20–25]. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1}, $n \geq 0$, of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by $u = \lim_{n\to\infty} u_n$.

3. Homotopy Perturbation Method (HPM) and He's Polynomials

To explain He's homotopy perturbation method, we consider a general equation of the type

$$L(u) = 0, \tag{3}$$

where L is any integral or differential operator. We define a convex homotopy $H(u, p)$ by

$$H(u,p) = (1-p)F(u) + pL(u), \tag{4}$$

where $F(u)$ is a functional operator with known solutions v_0, which can be obtained easily. It is clear that, for

$$H(u,p) = 0, \tag{5}$$

we have

$$H(u,0) = F(u), \quad H(u,1) = L(u).$$

This shows that $H(u,p)$ continuously traces an implicitly defined curve from a starting point $H(v_0,0)$ to a solution function $H(f,1)$. The embedding parameter monotonically increases from zero to unit as the trivial problem $F(u) = 0$ continuously deforms the original problem $L(u) = 0$. The embedding parameter $p \in (0,1]$ can be considered as an expanding parameter [12, 13, 16–22, 29–34]. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter [12, 13, 16–22] to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots. \tag{6}$$

If $p \to 1$, then (6) corresponds to (4) and becomes the approximate solution of the form

$$f = \lim_{p\to 1} u = \sum_{i=0}^{\infty} u_i. \tag{7}$$

It is well known that series (7) is convergent for most of the cases and also the rate of convergence is dependent on $L(u)$; see [12, 13, 16–22]. We assume that (7) has a unique solution. The comparisons of like powers of p give solutions of various orders. In sum, according to [12, 13], He's HPM considers the nonlinear term as

$$N(u) = \sum_{i=0}^{\infty} p^i H_i = H_0 + pH_1 + p^2 H_2 + \ldots,$$

where H_n are the so-called He's polynomials [12, 13], which can be calculated by using the formula

$$H_n(u_0,\ldots,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(N\left(\sum_{i=0}^{n} p^i u_i\right)\right)_{p=0}, \quad n = 0,1,2,\ldots.$$

4. Modified Variational Iteration Method (mVIM)

The modified variational iteration method (mVIM) [30, 32–34] is obtained by the elegant coupling of correction functional (2) of variational iteration method (VIM) with He's polynomials and is given by

$$\sum_{n=0}^{\infty} p^{(n)} u_n = u_0(x) + p\int_0^x \lambda(s)\left(\sum_{n=0}^{\infty} p^{(n)} L(u_n) + \sum_{n=0}^{\infty} p^{(n)} N(\tilde{u}_n)\right)\mathrm{d}s - \int_0^x \lambda(s) g(s)\mathrm{d}s. \tag{8}$$

Comparisons of like powers of p give solutions of various orders.

5. Numerical Applications

In this section, we apply the modified variational iteration method (mVIM) for solving integro-differential and coupled systems of integro-differential equations. Numerical results are very encouraging.

Example 5.1. Consider the following second-order system of nonlinear integro-differential equations:

$$u''(x) = 1 - \frac{1}{3}x^3 - \frac{1}{2}(v')^2 + \frac{1}{2}\int_0^x (u^2(t) + v^2(t))\mathrm{d}t,$$

$$v''(x) = -1 + x^2 - xu(x) + \frac{1}{4}\int_0^x (u^2(t) - v^2(t))\mathrm{d}t,$$

with the initial conditions

$$u(0)=1, \quad u'(0)=2, \quad v(0)=-1, \quad v'(0)=0.$$

The exact solution for this problem is

$$u(x)=x+\mathrm{e}^x, \quad v(x)=x-\mathrm{e}^x.$$

The correction functional is given by

$$u_{n+1}(x)=u_n(x)-\int_0^x \lambda(s)\left[\frac{\mathrm{d}^2u_n}{\mathrm{d}x^x}-\left(\left(1-\frac{1}{3}x^3-\frac{1}{2}\left(\tilde{v}_n'\right)^2\right)+\frac{1}{2}\int_0^x\left(\tilde{u}_n^2(t)+\tilde{v}_n^2(t)\right)\mathrm{d}t\right)\right]\mathrm{d}s,$$

$$v_{n+1}(x)=v_n(x)-\int_0^x \lambda(s)\left[\frac{\mathrm{d}^2v_n}{\mathrm{d}x^x}-\left(\left(1+x^2-x\tilde{u}_n(x)\right)+\frac{1}{2}\int_0^x\left(\tilde{u}_n^2(t)-\tilde{v}_n^2(t)\right)\mathrm{d}t\right)\right]\mathrm{d}s.$$

Making the above functional stationary, the Lagrange multiplier can be identified as $\lambda(s)=s-x$, and we get

$$u_{n+1}(x)=u_n(x)-\int_0^x (s-x)\left[\frac{\mathrm{d}^2u_n}{\mathrm{d}x^x}-\left(\left(1-\frac{1}{3}x^3-\frac{1}{2}(v_n')^2\right)+\frac{1}{2}\int_0^x(u_n^2(t)+v_n^2(t))\mathrm{d}t\right)\right]\mathrm{d}s,$$

$$v_{n+1}(x)=v_n(x)-\int_0^x (s-x)\left[\frac{\mathrm{d}^2v_n}{\mathrm{d}x^x}-\left((-1+x^2-xu_n(x))+\frac{1}{2}\int_0^x(u_n^2(t)-v_n^2(t))\mathrm{d}t\right)\right]\mathrm{d}s.$$

Applying the modified variational iteration method (mVIM), we obtain

$$u_0+pu_1+\cdots=1+2x+p\int_0^x(s-x)\left[\left(\frac{\mathrm{d}^2u_0}{\mathrm{d}x^x}+p\frac{\mathrm{d}^2u_1}{\mathrm{d}x^x}+p^2\frac{\mathrm{d}^2u_2}{\mathrm{d}x^x}+\cdots\right)-\left(1-\frac{1}{3}x^3-\frac{1}{2}(v_0'+pv_1'+\cdots)^2\right)\right]\mathrm{d}s$$
$$-\frac{1}{2}p\int_0^x(s-x)((u_0^2+pu_1^2+\cdots)+(v_0^2+pv_1^2+\cdots))\mathrm{d}s,$$

$$v_0+pv_1+\cdots=-1+p\int_0^x(s-x)\left[\left(\frac{\mathrm{d}^2v_0}{\mathrm{d}x^x}+p\frac{\mathrm{d}^2v_1}{\mathrm{d}x^x}+p^2\frac{\mathrm{d}^2v_2}{\mathrm{d}x^x}+\cdots\right)-(-1+x^2-x(u_0+pu_1+\cdots))\right]\mathrm{d}s$$
$$-\frac{1}{2}p\int_0^x(s-x)((u_0^2+pu_1^2+\cdots)-(v_0^2+pv_1^2+\cdots))\mathrm{d}s.$$

Comparing the co-efficients of like powers of p, following approximations are obtained:

$$p^{(0)}:\ u_0(x)=1+2x, \quad v_0(x)=-1,$$

$$p^{(1)}:\ u_1(x)=1+2x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{12}x^4+\frac{1}{60}x^5,$$
$$v_1(x)=-1-\frac{1}{2}x^2-\frac{1}{6}x^3-\frac{1}{24}x^4+\frac{1}{60}x^5,$$

$$p^{(2)}:\ u_2(x)=1+2x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{12}x^4+\frac{1}{60}x^5-\frac{1}{24}x^4-\frac{1}{120}x^5+\frac{1}{720}x^6+\frac{17}{5040}x^7+\frac{1}{672}x^8$$
$$+\frac{53}{120960}x^9-\frac{1}{103680}x^{10}+\frac{17}{228096}x^{11}+\frac{1}{1900800}x^{12}+\frac{1}{6177600}x^{13},$$

$$v_2(x)=-1-\frac{1}{2}x^2-\frac{1}{6}x^3-\frac{1}{24}x^4+\frac{1}{60}x^5+\frac{1}{120}x^5-\frac{1}{720}x^6-\frac{11}{10080}x^7+\frac{13}{241920}x^9+\frac{17}{1036800}x^{10}$$
$$+\frac{47}{11404800}x^{11}+\frac{1}{1267200}x^{12},$$

$$\vdots$$

The series solution is given by

$$u(x)=1+2x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{12}x^4+\frac{1}{60}x^5-\frac{1}{24}x^4-\frac{1}{120}x^5+\frac{1}{720}x^6+\frac{17}{5040}x^7+\frac{1}{672}x^8+\frac{53}{120960}x^9$$
$$-\frac{1}{103680}x^{10}+\frac{17}{228096}x^{11}+\frac{1}{1900800}x^{12}+\frac{1}{6177600}x^{13}+\cdots,$$

$$v(x)=-1-\frac{1}{2}x^2-\frac{1}{6}x^3-\frac{1}{24}x^4+\frac{1}{60}x^5\frac{1}{120}x^5-\frac{1}{720}x^6-\frac{11}{10080}x^7+\frac{13}{241920}x^9+\frac{17}{1036800}x^{10}$$
$$+\frac{47}{11404800}x^{11}+\frac{1}{1267200}x^{12}+\cdots.$$

Table 1. Error estimates.

x	Errors* U	V
−1.0	2.16E-03	8.74E-04
−0.8	4.80E-04	1.85E-04
−0.6	6.87E-05	2.49E-05
−0.4	4.35E-06	1.46E-06
−0.2	3.71E-08	1.15E-08
0.0	0.00000	0.00000
0.2	4.50E-08	1.10E-08
0.4	6.27E-06	1.42E-06
0.6	1.17E-04	2.38E-05
0.8	9.69E-04	1.73E-04
1.0	5.06E-03	7.89E-04

*Error = Exact solution – Series solution.

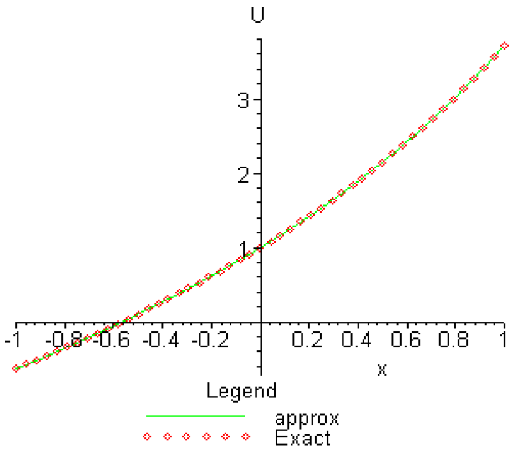

Fig. 1.

Example 5.2. Consider the following second-order system of nonlinear integro-differential equations:

$$u''(x) = x + 2x^3 + 2(v'(x))^2 - \int_0^x ((v'(t))^2 + u(t)w''(t)v^2(t))\mathrm{d}t,$$

$$v''(x) = 3x^2 - xu(x) + \frac{1}{4}\int_0^x (txv'(t)u''(t) - w'(t))\mathrm{d}t,$$

$$w''(x) = 2 - \frac{4}{3}x^3 + (u''(x))^2 - 2u^2(x) + \int_0^x x(v^2(t) + (u'(t))^2 + t^3 w''(t))\mathrm{d}t,$$

with initial conditions

$$u(0) = 1, \quad u'(0) = 0, \quad v(0) = 0, \quad v'(0) = 1, \quad w(0) = 0, \quad w'(0) = 0.$$

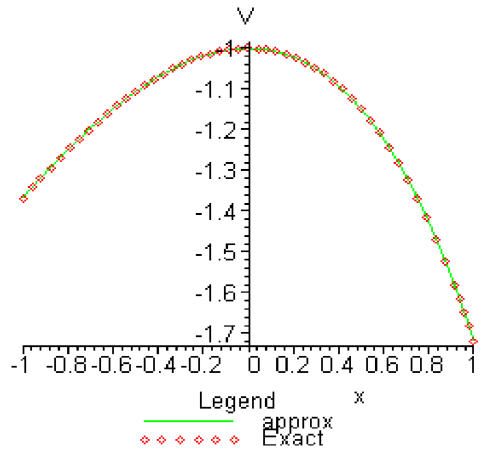

Fig. 2.

The exact solutions for this problem is

$$u(x) = x^2, \quad v(x) = x, \quad w(x) = 3x^2.$$

The correction functional is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-x)\Bigg[\frac{\mathrm{d}^2 u_n}{\mathrm{d}x^2} - \Big(x + 2x^3 + 2\left(\tilde{v}_n'(x)\right)^2\Big) + \int_0^x \left(\tilde{v}_n''(t)\right)^2 + \tilde{u}_n(t)\tilde{w}_n''(t)\tilde{v}_n^2(t)\mathrm{d}t\Bigg]\mathrm{d}s,$$

$$v_{n+1}(x) = v_n(x) + \int_0^x (s-x)\Bigg[\frac{\mathrm{d}^2 v_n}{\mathrm{d}x^2} - \left(3x^2 - x\tilde{u}_n(x)\right) - \frac{1}{4}\int_0^x \left(\left(tx\tilde{v}_n'(t)\tilde{u}_n''(t) - \tilde{w}_n'(t)\right)\right)\mathrm{d}t\Bigg]\mathrm{d}s,$$

$$w_{n+1}(x) = w_n(x) + \int_0^x (s-x)\Bigg[\frac{\mathrm{d}^2 w_n}{\mathrm{d}x^2} - \Big(2 - \frac{4}{3}x^3 + \left(\tilde{u}_n''(x)\right)^2 - 2\tilde{u}_n^2(x)\Big) - \int_0^x x\left(\tilde{v}_n^2(t) + \left(\tilde{u}_n'(t)\right)^2 + t^3\tilde{w}_n''(t)\right)\mathrm{d}t\Bigg]\mathrm{d}s.$$

Making the above functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = s - x$, and we get

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-x)\Bigg[\frac{\mathrm{d}^2 u_n}{\mathrm{d}x^2} - (x + 2x^3 + 2(v_n(x))^2) + \int_0^x (v_n''(t))^2 + u_n(t)w_n''(t)v_n^2(t)\mathrm{d}t\Bigg]\mathrm{d}s,$$

$$v_{n+1}(x)=v_n(x)+\int_0^x(s-x)\left[\frac{\mathrm{d}^2v_n}{\mathrm{d}x^2}-(3x^2-xu_n(x))-\frac{1}{4}\int_0^x((txv_n'(t)u_n''(t)-w_n'(t)))\mathrm{d}t\right]\mathrm{d}s,$$

$$w_{n+1}(x)=w_n(x)+\int_0^x(s-x)\left[\frac{\mathrm{d}^2w_n}{\mathrm{d}x^2}-\left(2-\frac{4}{3}x^3+(u_n''(x))^2-2u_n^2(x)\right)-\int_0^x x(v_n^2(t)+(u_n'(t))^2+t^3w_n''(t))\mathrm{d}t\right]\mathrm{d}s.$$

Applying the modified variational iteration method (mVIM), we obtain

$$u_0+pu_1+\cdots=1+p\int_0^x(s-x)\left(\frac{\mathrm{d}^2u_0}{\mathrm{d}x^x}+p\frac{\mathrm{d}^2u_1}{\mathrm{d}x^x}+\cdots\right)-\left((x+2x^3+2(v_0'+pv_1'+\cdots)^2)\right.$$

$$\left.+\int_0^x(v_0'+pv_1'+\cdots)^2\mathrm{d}t\right)\mathrm{d}s-p\int_0^x(s-x)((u_0+pu_1+\cdots)(v_0+pv_1+\cdots)^2(w_0''+pw_1''+\cdots))\mathrm{d}s,$$

$$v_0+pv_1+\cdots=x+p\int_0^x(s-x)\left(\frac{\mathrm{d}^2v_0}{\mathrm{d}x^x}+p\frac{\mathrm{d}^2v_1}{\mathrm{d}x^x}+\cdots\right)-((3x^2-x(u_0+pu_1+\cdots)))\mathrm{d}s$$

$$-\frac{1}{4}p\int_0^x(s-x)(sx(v_0'+pv_1'+\cdots)(u_0^2+pu_1^2+\cdots)(u_0''+pu_1''+\cdots))+p\int_0^x(x-s)(w_0'+pw_1'+\cdots)\mathrm{d}s,$$

$$w_0+pw_1+\cdots=p\int_0^x(s-x)\left[\left(\frac{\mathrm{d}^2w_0}{\mathrm{d}x^x}+p\frac{\mathrm{d}^2w_1}{\mathrm{d}x^x}+\cdots\right)-\left(2-\frac{4}{3}x^3+(u_0''+pu_1''+\cdots)^2-2(u_0+pu_1+\cdots)^2\right)\right]\mathrm{d}s$$

$$-p\int_0^x x(x-s)((u_0'+pu_1'+\cdots)^2+t^3(w_0''+pw_1''+\cdots))\mathrm{d}s-p\int_0^x x(v_0+pv_1+\cdots)^2\mathrm{d}s.$$

Comparing the co-efficient of like powers of p, following approximations are obtained:

$$p^{(0)}:\ u_0(x)=1,\quad v_0(x)=x,\quad w_0(x)=0,$$

$$p^{(1)}:\ u_1(x)=1+x^2+\frac{1}{10}x^5,$$
$$v_1(x)=x+\frac{1}{4}x^4,$$
$$w_1(x)=x^2-\frac{1}{15}x^5+\frac{1}{60}x^6,$$

$$p^{(2)}:\ u_2(x)=1+x^2+\frac{1}{10}x^5+\frac{1}{6}x^5-\frac{1}{60}x^6+\frac{197}{5040}x^8-\frac{1}{336}x^9+\frac{1}{7425}x^{11}-\frac{17}{26400}x^{12},$$
$$v_2(x)=x+\frac{1}{4}x^4-\frac{1}{12}x^4-\frac{1}{630}x^7-\frac{41}{3360}x^8+\frac{1}{440}x^{11},$$
$$w_2(x)=x^2-\frac{1}{15}x^5+\frac{1}{60}x^6-\frac{1}{20}x^6+\frac{13}{168}x^8-\frac{227}{30240}x^9+\frac{1}{1440}x^{10}+\frac{1}{3960}x^{11}-\frac{1}{6600}x^{12},$$
$$\vdots$$

The series solution is given by

$$u(x)=1+x^2+\frac{1}{10}x^5+\frac{1}{6}x^5-\frac{1}{60}x^6+\frac{197}{5040}x^8-\frac{1}{336}x^9+\frac{1}{7425}x^{11}-\frac{17}{26400}x^{12}+\cdots,$$
$$v(x)=x+\frac{1}{4}x^4-\frac{1}{12}x^4-\frac{1}{630}x^7-\frac{41}{3360}x^8+\frac{1}{440}x^{11}+\cdots,$$
$$w(x)=x^2-\frac{1}{15}x^5+\frac{1}{60}x^6-\frac{1}{20}x^6+\frac{13}{168}x^8-\frac{227}{30240}x^9+\frac{1}{1440}x^{10}+\frac{1}{3960}x^{11}-\frac{1}{6600}x^{12}+\cdots.$$

Example 5.3. Consider the two-dimensional nonlinear inhomogeneous initial boundary value problem for the integro-differential equation related to the Blasius problem:

$$y''(x)=\alpha-\frac{1}{2}\int_0^x y(t)y''(t)\mathrm{d}t,\quad -\infty<x<0,$$

with the boundary conditions

$$y(0)=0,\quad y'(0)=1,$$

Table 2. Error estimates.

x	Errors* U	V	W
−1.0	2.33E-02	3.48E-02	3.48E-01
−0.8	4.79E-03	1.25E-03	1.25E-01
−0.6	5.77E-04	3.12E-04	3.12E-02
−0.4	2.83E-05	4.17E-05	4.17E-03
−0.2	1.72E-07	1.29E-07	1.29E-04
0.0	0.00000	0.00000	0.00000
0.2	7.20E-08	1.26E-08	1.26E-04
0.4	2.80E-06	4.00E-06	4.00E-03
0.6	7.15E-05	3.07E-05	3.07E-02
0.8	1.64E-03	1.34E-03	1.34E-01
1.0	1.53E-02	4.37E-02	4.37E-01

*Error = Exact solution – Series solution.

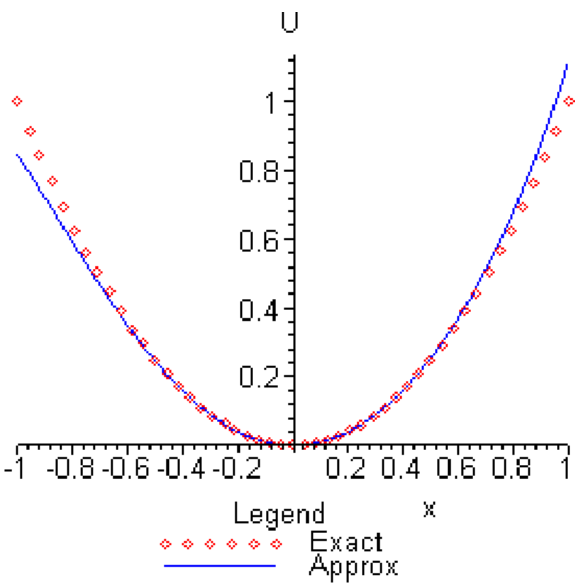

Fig. 3.

and

$$\lim_{x\to\infty} y'(x) = 0,$$

where constant α is positive and defined by

$$y''(0) = \alpha \quad \alpha > 0.$$

The correction functional is given by

$$y_{n+1}(x) = x + \int_0^x (s-x)\left[\frac{\mathrm{d}^2 y_n}{\mathrm{d}x^2} - \left(\alpha - \frac{1}{2}\int_0^x \left(\tilde{y}_n(s)\tilde{y}_n''(s)\right)\mathrm{d}s\right)\right]\mathrm{d}s, \; -\infty < x < 0.$$

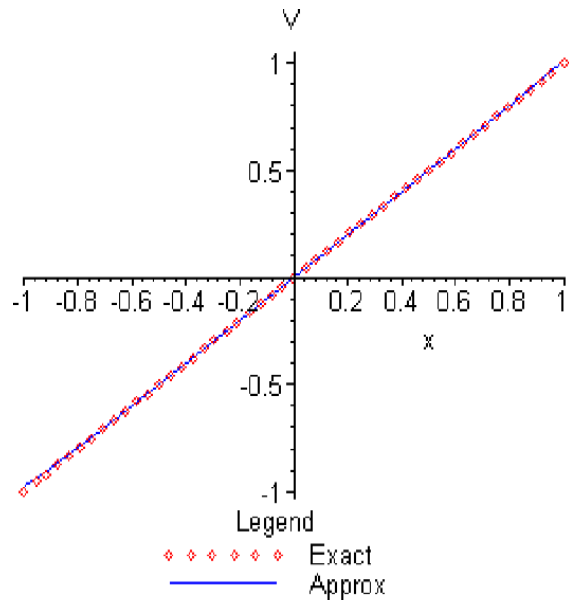

Fig. 4.

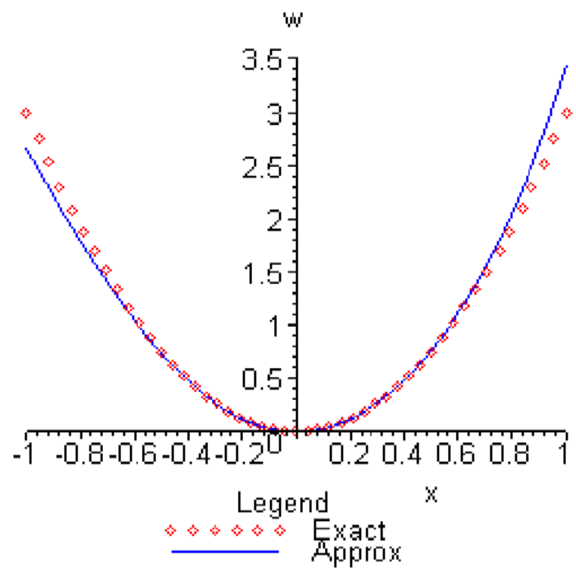

Fig. 5.

Making the above functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = s - x$, and we get

$$y_{n+1}(x) = x + \int_0^x (s-x)\left[\frac{\mathrm{d}^2 y_n}{\mathrm{d}x^2} - \left(\alpha - \frac{1}{2}\int_0^x (y_n(s)y_n''(s))\mathrm{d}s\right)\right]\mathrm{d}s, \; -\infty < x < 0.$$

Table 3. Padé approximants and numerical value of α.

Padé approximant	α
[2/2]	0.5778502691
[3/3]	0.5163977793
[4/4]	0.5227030798

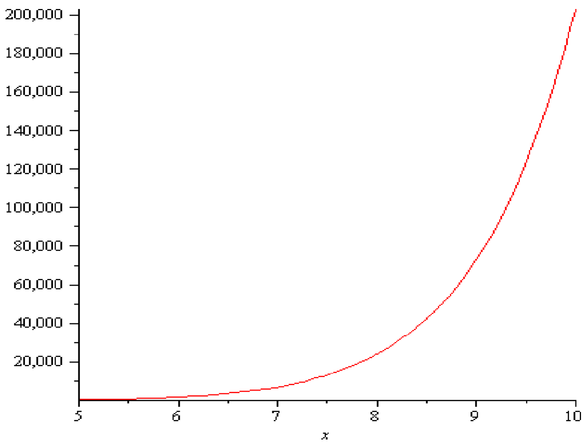

Fig. 6. Numerical value $\alpha = 0.5778502691$.

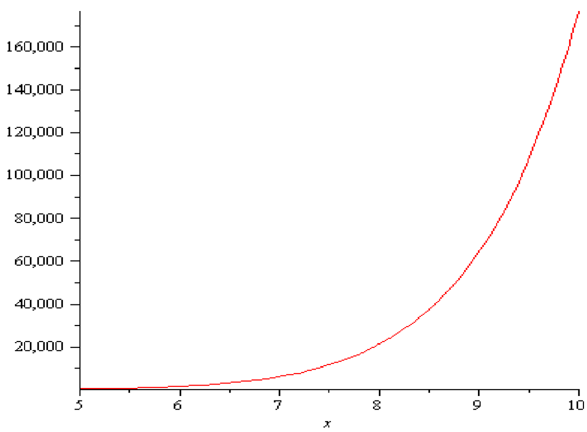

Fig. 7. Numerical value $\alpha = 0.5163977793$.

Applying the modified variational iteration method (mVIM), we obtain

$$y_0 + py_1 + p^2y_2 + \cdots = x + p\int_0^x (s-x)\left(\frac{\mathrm{d}^2y_0}{\mathrm{d}x^2} + p\frac{\mathrm{d}^2y_1}{\mathrm{d}x^2} + \cdots - \left(\int_0^x (y_0 + py_1 + \cdots)(y_0'' + py_1'' + \cdots)\mathrm{d}s\right)\mathrm{d}s.$$

Proceeding as before, the series solution is given as

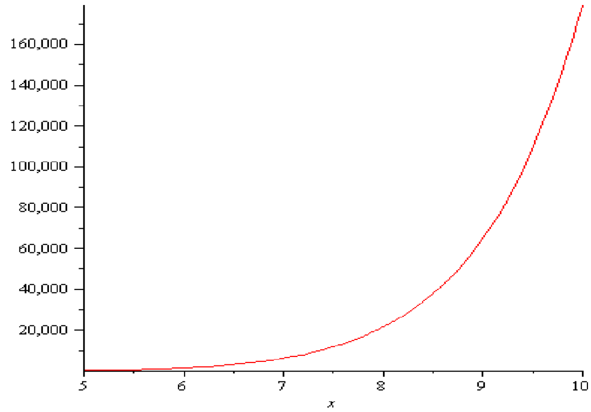

Fig. 8. Numerical value $\alpha = 0.5227030798$.

$$\begin{aligned} y(x) &= x + \frac{1}{2}\alpha x^2 - \frac{1}{48}\alpha x^4 - \frac{1}{240}\alpha^2 x^5 + \frac{1}{960}\alpha x^6 \\ &+ \frac{11}{20160}\alpha^2 x^7 + \left(\frac{11}{161280}\alpha^3 + \frac{1}{960}\alpha\right)x^8 \\ &- \frac{43}{967680}\alpha^2 x^9 + \left(\frac{1}{52960}\alpha - \frac{5}{387072}\alpha^3\right)x^{10} \\ &+ \left(\frac{587}{212889600}\alpha^2 - \frac{5}{4257792}\alpha^4\right)x^{11} \\ &+ \left(-\frac{1}{16220160}\alpha + \frac{1}{7257792}\alpha^3\right)x^{12} + \cdots, \end{aligned}$$

and consequently

$$\begin{aligned} y'(x) &= 1 + \alpha x - \frac{1}{12}\alpha x^3 - \frac{1}{48}\alpha^2 x^4 + \frac{1}{160}\alpha x^5 \\ &+ \frac{11}{2880}\alpha^2 x^6\left(\frac{11}{20160}\alpha^3 - \frac{1}{2688}\alpha\right)x^7 \\ &- \frac{43}{107520}\alpha^2 x^8 + 10\left(\frac{1}{552960}\alpha - \frac{5}{387072}\alpha^3\right)x^9 \\ &+ 11\left(\frac{587}{212889600}\alpha^2 - \frac{5}{4257792}\alpha^4\right)x^{10} \\ &+ 12\left(-\frac{1}{16220160}\alpha + \frac{1}{725760}\alpha^3\right)x^{11} + \cdots. \end{aligned}$$

The diagonal Padé approximants can be applied to determine a numerical value for the constant α by using the given condition.

6. Conclusion

In this paper, we applied a modified variational iteration method (mVIM) for solving integro-differential equations and coupled systems of integro-differential equations. The proposed technique is employed without using linearization, discretization or restrictive assumptions. Moreover, the suggested method is free from round off errors and calculation of the so-called Adomian's polynomials. It may be concluded that the mVIM is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems.

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