Variational Iteration Method for Burgers' and Coupled Burgers' Equations Using He's Polynomials

Syed Tauseef Mohyud-Din[a], Muhammad Aslam Noor[b], and Khalida Inayat Noor[b]

[a] HITEC University, Taxila Cantt. Pakistan
[b] Department of Mathematics, COMSATS Institute of Information Technology, Islamabad Pakistan

Reprint requests to S. T. M.-D.; E-mail: syedtauseefs@hotmail.com

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In this paper, we apply a modified version of the variational iteration method (MVIM) for solving Burgers' and coupled Burgers' equations. The proposed modification is made by introducing He's polynomials in the correction functional of the variational iteration method (VIM). The use of Lagrange multiplier coupled with He's polynomials are the clear advantages of this technique over the decomposition method.

Key words: Modified Variational Iteration Method; He's Polynomials; Burgers' Equations.

1. Introduction

Burgers' and coupled Burgers' equations are of great significance in the diversified physical problems related to engineering and applied sciences. The applications of these equations include approximation theory of flow through a shock wave travelling in a viscous fluid and in Burgers' model of turbulence [1]. Several techniques including decomposition, finite element, Galerkin and cubic spline are employed to solve such equations analytically and numerically, see [1] and the reference therein. Most of these used schemes are coupled with the inbuilt deficiencies like calculation of the so-called Adomian's polynomials and non-compatibility with the physical nature of the problems. He developed the variational iteration and homotopy perturbation methods and applied them on a wide class of physical problems, see [2 – 14] and the references therein. In a later work Ghorbani et al. introduced He's polynomials [15, 16] which are compatible with Adomian's polynomials but are easier to calculate and are more user friendly [2, 3]. It has been established by Xu [17], Shou and He [18] that He's variational iteration method (VIM) has clear advantages over the decomposition method. It needs to be highlighted that He's variational iteration method (VIM) has become the hottest topic for applied sciences [19]. Recently, Noor and Mohyud-Din [20 – 25] made the elegant coupling of He's polynomials and the correction functional of VIM. It is worth mentioning that He's polynomials are calculated by applying He's homotopy perturbation method (HPM) [2 – 8, 15, 16, 18, 20 – 28]. This very reliable modified version [20 – 25] (MVIM) has been proved useful in coping with the physical nature of the nonlinear problems and hence absorbs all the positive features of the coupled techniques. The basic motivation of this paper is the application of this elegant coupling of He's polynomials and correction functional of VIM for solving Burgers' and coupled Burgers' equations. The numerical results are very encouraging.

1.1. Variational Iteration Method Using He's Polynomials (MVIM)

To illustrate the basic concept of the MVIM, we consider the following general differential equation:

$$Lu + Nu = g(x), \tag{1}$$

where L is a linear operator, N a nonlinear operator, and g(x) is the forcing term. According to VIM [2, 4, 9 – 14, 17, 18, 20 – 25], we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\big[Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\big]\mathrm{d}\xi, \tag{2}$$

where λ is a Lagrange multiplier [2, 4, 9 – 14], $\tilde{u}_n$ is a restricted variation; (2) is called a correction func-

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tional. Now, we apply He's polynomials [15, 16]

$$\sum_{n=0}^{\infty} p^{(n)} u_n = u_0(x) + p\int_0^x \lambda(\xi)\left[\sum_{n=0}^{\infty} p^{(n)} L(u_n) + \sum_{n=0}^{\infty} p^{(n)} N(\tilde{u}_n)\right]\mathrm{d}\xi - \int_0^x \lambda(\xi)\, g(\xi)\mathrm{d}\xi, \quad (3)$$

which is the MVIM [20 – 25] and is formulated by the coupling of VIM and He's polynomials. The comparison of like powers of p gives solutions of various orders.

2. Numerical Applications

In this section, we apply the variational iteration method using He's polynomials (MVIM) for solving Burgers' and coupled Burgers' equations.

Example 3.1 [1] Consider the following one-dimensional Burgers equation:

$$u_t + uu_{xx} - vu_{xx} = 0,$$

with the initial condition

$$u(x,0) = \frac{\alpha+\beta+(\beta-\alpha)\exp(\gamma)}{1+\exp(\gamma)}, \quad t \ge 0,$$

where $\gamma = (\alpha/v(x-\lambda)$ and the parameters α, β, λ are arbitrary constants. The correction functional is given by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s)\left(\frac{\partial u_n}{\partial t} + \tilde{u}_n \frac{\partial^2 \tilde{u}_n}{\partial^2 x} - v\frac{\partial^2 \tilde{u}_n}{\partial x^2}\right)\mathrm{d}s.$$

Making the above functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = -1$ and we get

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n}{\partial t} + u_n \frac{\partial^2 u_n}{\partial x^2} - v\frac{\partial^2 u_n}{\partial x^2}\right)\mathrm{d}s.$$

Now we apply the variational iteration method using He's polynomials (MVIM):

$$u_0 + pu_1 + p^2 u_2 + \cdots = \frac{\alpha+\beta+(\beta-\alpha)\exp(\gamma)}{1+\exp(\gamma)} - p\int_0^t \left[\frac{\partial u_0}{\partial t} + p\frac{\partial u_1}{\partial t} + p^2\frac{\partial u_2}{\partial t} + \cdots\right]\mathrm{d}s - p\int_0^t \left[(u_0 + pu_1 + p^2u_2 + \cdots)\cdot\left(\frac{\partial^2 u_0}{\partial x^2} + p\frac{\partial^2 u_1}{\partial x^2} + \cdots\right) - v\left(\frac{\partial^2 u_0}{\partial x^2} + p\frac{\partial^2 u_1}{\partial x^2} + \cdots\right)\right]\mathrm{d}s.$$

By comparing the co-efficient of like powers of p, we obtain

$$p^{(0)} : u_0(x,t) = \frac{\alpha+\beta+(\beta-\alpha)\exp(\gamma)}{1+\exp(\gamma)},$$

$$p^{(1)} : u_1(x,t) = \frac{\alpha+\beta+(\beta-\alpha)\exp(\gamma)}{1+\exp(\gamma)} + \frac{2\alpha\beta^2\exp(\gamma)}{v(1+\exp(\gamma))}t,$$

$$p^{(2)} : u_2(x,t) = \frac{\alpha+\beta+(\beta-\alpha)\exp(\gamma)}{1+\exp(\gamma)} + \frac{2\alpha\beta^2\exp(\gamma)}{v(1+\exp(\gamma))}t + \frac{\alpha^3\beta^2\exp(\gamma)(-1+\exp(\gamma))}{v^2(1+\exp(\gamma))^3}t^3,$$

$$\vdots$$

The series solution is therefore given by

$$u(x,t) = \frac{\alpha+\beta+(\beta-\alpha)\exp(\gamma)}{1+\exp(\gamma)} + \frac{2\alpha\beta^2\exp(\gamma)}{v(1+\exp(\gamma))}t + \frac{\alpha^3\beta^2\exp(\gamma)(-1+\exp(\gamma))}{v^2(1+\exp(\gamma))^3}t^3 + \frac{\alpha^4\beta^3\exp(\gamma)(1-4\exp(\gamma)+\exp(\gamma)^2)}{3v^3(1+\exp(\gamma))^4}t^4 + \cdots$$

and in a closed form by

$$u(x,t) = \frac{\alpha+\beta+(\beta-\alpha)\exp(x/v)(x-\beta t-\lambda)}{1+\exp(x/v)(x-\beta t-\lambda)}.$$

Example 3.2 [1] Consider the following homogeneous coupled Burgers equation:

$$u_t - u_{xx} - 2uu_x + (uv)_x = 0,$$
$$v_t - v_{xx} - 2vv_x + (uv)_x = 0$$

with the initial conditions

$$u(x,0) = \sin x, \quad v(x,0) = \sin x.$$

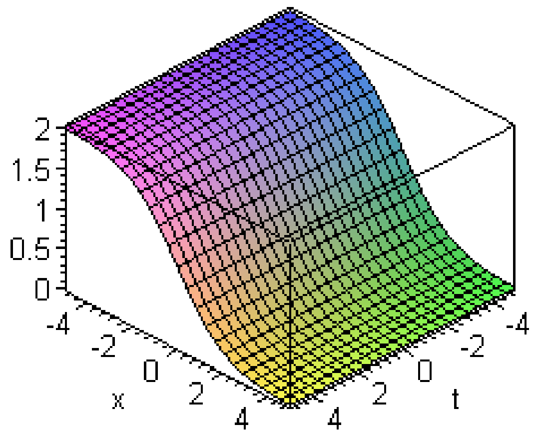

Fig. 1. Solution $u(x,t)$ with the parameters $\alpha = \beta = v = \lambda = 1$.

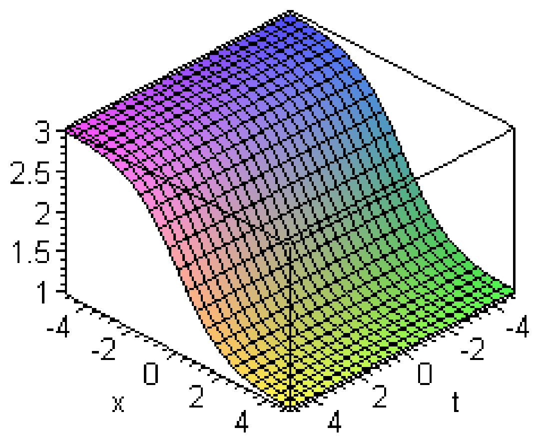

Fig. 2. Solution $u(x,t)$ with the parameters $\alpha = v = 1$, $\beta = \lambda = 2$.

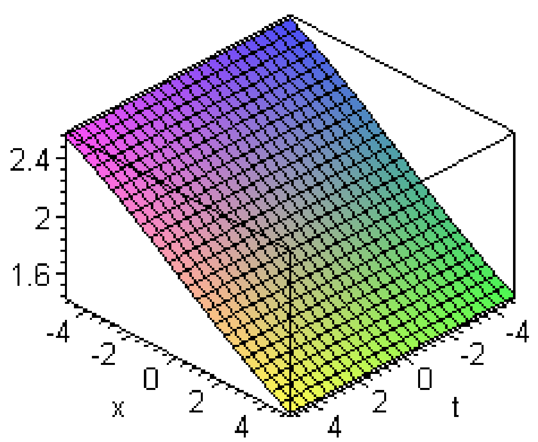

Fig. 3. Solution $u(x,t)$ with the parameters $\alpha = 1$, $\beta = 2$, $v = 4$, $\lambda = 3$.

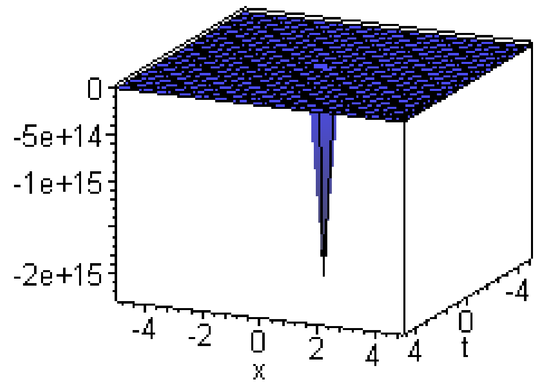

Fig. 4. Solution $u(x,t)$ with the parameters $\alpha = 1$, $\beta = 0.2$, $v = 0.5$, $\lambda = 1$.

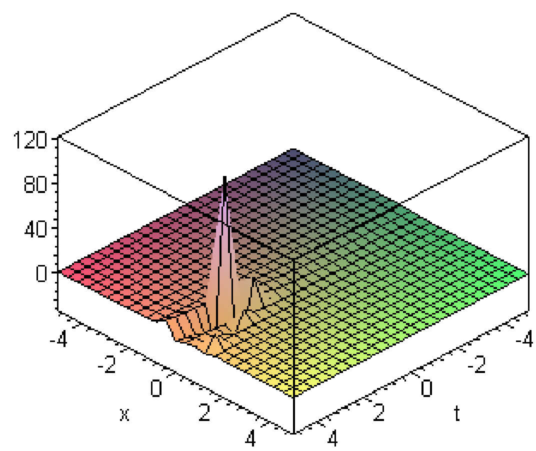

Fig. 5. Solution $u(x,t)$ with the parameters $\alpha = 1$, $\beta = 0.2$, $v = 0.5$, $\lambda = 0.4$.

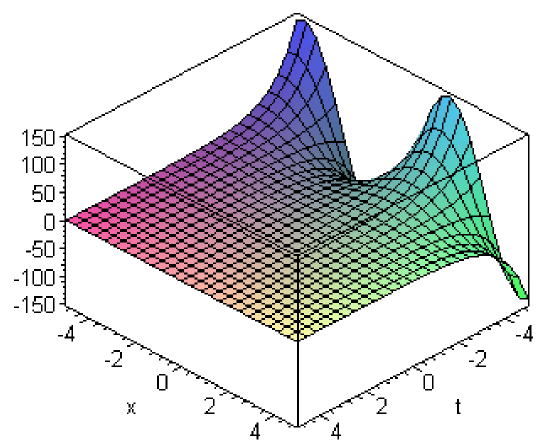

Fig. 6. Plot of the solutions $u(x,t)$ or $v(x,t)$.

The correction functional for the above mentioned coupled system with Lagrange multiplier $\lambda(s) = -1$, is given by

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n}{\partial t} - \frac{\partial^2 u_n}{\partial x^2} - 2u_n(u_n)_x + (u_n v_n)_x \right) \mathrm{d}s.$$

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \left(\frac{\partial v_n}{\partial t} - \frac{\partial^2 v_n}{\partial x^2} - 2v_n(v_n)_x + (u_n v_n)_x \right) \mathrm{d}s.$$

Now we apply the variational iteration method using He's polynomials (MVIM):

$$u_0 + pu_1 + p^2u_2 + \cdots = u_0(x,t) + p\int_0^t \lambda(s)\left[\left(\frac{\partial u_0}{\partial t} + p\frac{\partial u}{\partial t} + \cdots\right) - \left(\frac{\partial^2 u_0}{\partial x^2} + p\frac{\partial^2 u_1}{\partial x^2} + p^2\frac{\partial^2 u_2}{\partial x^2} + \cdots\right)\right]\mathrm{d}s$$
$$-p\int_0^t \left[2(u_0 + pu_1 + \cdots)\left(\frac{\partial u_0}{\partial x} + p\frac{\partial u_1}{\partial x} + \cdots\right) - ((u_0 + pu_1 + \cdots)(v_0 + pv_1 + \cdots))_x\right]\mathrm{d}s,$$
$$v_0 + pv_1 + p^2v_2 + \cdots = v_0(x,t) + p\int_0^t \lambda(s)\left[\left(\frac{\partial v_0}{\partial t} + p\frac{\partial v_1}{\partial t} + \cdots\right) - \left(\frac{\partial^2 v_0}{\partial x^2} + p\frac{\partial^2 v_1}{\partial x^2} + p^2\frac{\partial^2 v_2}{\partial x^2} + \cdots\right)\right]\mathrm{d}s$$
$$-p\int_0^t \left[2(v_0 + pv_1 + \cdots)\left(\frac{\partial v_0}{\partial x} + p\frac{\partial v_1}{\partial x} + \cdots\right) - ((u_0 + pu_1 + \cdots)(v_0 + pv_1 + \cdots))_x\right]\mathrm{d}s.$$

By comparing the co-efficient of like powers of p, we obtain

$$p^{(0)}: \begin{cases} u_0(x,t) = \sin x, \\ v_0(x,t) = \sin x, \end{cases}$$

$$p^{(1)}: \begin{cases} u_1(x,t) = \sin x - t\sin x, \\ v_0(x,t) = \sin x - t\sin x, \end{cases}$$

$$p^{(2)}: \begin{cases} u_2(x,t) = \sin x - t\sin x + \frac{t^2}{2!}\sin x, \\ v_2(x,t) = \sin x - t\sin x + \frac{t^2}{2!}\sin x, \end{cases}$$

$\vdots$

The series solutions are given by

$$u(x,t) = \sin x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right),$$

$$v(x,t) = \sin x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right),$$

and the closed form solutions are given as

$$(u,v) = (\exp(-t)\sin x, \exp(-t)\sin x).$$

3. Conclusion

In this paper, we applied the variational iteration method using He's polynomials (MVIM) for finding the solutions of Burgers' and coupled Burgers' equations. The use of Lagrange multiplier coupled with He's polynomials are the clear advantages of this technique over the decomposition method.

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[1] M. A. Abdou and A. A. Soliman, J. Comput. Appl. Math. **181**, 245 (2005).
[2] J. H. He, Int. J. Mod. Phys. B **22**, 3487 (2008).
[3] J. H. He, Top. Meth. Nonlinear Anal. **31**, 205 (2008).
[4] J. H. He, Int. J. Mod. Phys. **10**, 1144 (2006).
[5] J. H. He, Appl. Math. Comput. **156**, 527 (2004).
[6] J. H. He, Int. J. Nonlinear Sci. Numer. Simul. **6**, 207 (2005).
[7] J. H. He, Appl. Math. Comput. **151**, 287 (2004).
[8] J. H. He, Int. J. Nonlinear Mech. **35**, 115 (2000).
[9] J. H. He, J. Comput. Appl. Math. **207**, 3 (2007).
[10] J. H. He and X. Wu, Comput. Math. Appl. **54**, 881 (2007).
[11] J. H. He, Internat. J. Nonlinear Mech. **34**, 699 (1999).
[12] J. H. He, Appl. Math. Comput. **114**, 115 (2000).
[13] J. H. He and X. H. Wu, Chaos, Solitons, and Fractals, **29**, 108 (2006).
[14] J. H. He, Phys. Scr. **76**, 680 (2007).
[15] A. Ghorbani and J. S. Nadjfi, Int. J. Nonlinear Sci. Numer. Simul. **8**, 229 (2007).
[16] A. Ghorbani, Chaos, Solitons, and Fractals, **39**, 1486 (2009).
[17] L. Xu, Comput. Math. Appl. **54**, 1071 (2007).
[18] D. H. Shou and J. H. He, Phys. Lett. A **372**, 233 (2008).
[19] S. Mitton, Science Watch, http://sciencewatch.com/ana/hot/phy/08julaug-phy/
[20] M. A. Noor and S. T. Mohyud-Din, Int. J. Nonlinear Sci. Numer. Simul. **9**, 141 (2008).
[21] M. A. Noor and S. T. Mohyud-Din, J. Appl. Math. Comput. **29**, 81 (2009).
[22] M. A. Noor and S. T. Mohyud-Din, Acta Applnda. Mathmtce. **104**, 257 (2008).
[23] M. A. Noor and S. T. Mohyud-Din, Math. Prob. Eng. 2008, Article ID 954794.
[24] M. A. Noor and S. T. Mohyud-Din, Math. Prob. Eng. 2008, Article ID 696734.
[25] M. A. Noor and S. T. Mohyud-Din, Comput. Math. Appl. **58**, 2182 (2009).
[26] S. T. Mohyud-Din, M. A. Noor, and K. I. Noor, Int. J. Nonlinear Sci. Numer. Simul. **10**, 223 (2009).
[27] M. A. Noor and S. T. Mohyud-Din, Int. J. Nonlinear Sci. Numer. Simul. **9**, 395 (2008).
[28] L. Xu, Comput. Math. Appl. **54**, 1067 (2007).
[29] L. Xu, Chaos, Solitons, and Fractals **39**, 1386 (2009).