Magneto Thermal Convection in a Compressible Couple-Stress Fluid

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The problem of thermal instability of compressible, electrically conducting couple-stress fluids in the presence of a uniform magnetic field is considered. Following the linearized stability theory and normal mode analysis, the dispersion relation is obtained. For stationary convection, the compressibility, couple-stress, and magnetic field postpone the onset of convection. Graphs have been plotted by giving numerical values of the parameters to depict the stability characteristics. The principle of exchange of stabilities is found to be satisfied. The magnetic field introduces oscillatory modes in the system that were non-existent in its absence. The case of overstability is also studied wherein a sufficient condition for the non-existence of overstability is obtained.

Key words: Thermal Instability; Compressible Couple-Stress Fluid; Uniform Magnetic Field; Linearized Theory; Normal Mode Analysis.

1. Introduction

The thermal instability of a fluid layer heated from below plays an important role in geophysics, oceanography, atmospheric physics, etc., and has been investigated by many authors, e. g. Bénard [1], Rayleigh [2], and Jeffreys [3]. A detailed account of the theoretical and experimental studies of the so called 'Bénard convection' in Newtonian fluids has been given by Chandrasekhar [4]. The Boussinesq approximation, which states that the density can be treated as a constant in all terms of the equations of motion except the external force term has been used throughout. Sharma [5] has considered the effect of rotation and magnetic field on the thermal instability in compressible fluids. The fluid has been considered to be Newtonian in all the above studies.

With the growing importance of non-Newtonian fluids in modern technology and industry, the investigations of such fluids are desirable. The presence of small amounts of additives in a lubricant can improve bearing performance by increasing the lubricant viscosity and thus producing an increase in the load capacity. These additives in a lubricant also reduce the coefficient of friction and increase the temperature range in which the bearing can operate. A number of theories of the microcontinuum have been postulated and applied (Stokes [6]; Lai et al. [7]; Walicka [8]). The theory of couple-stress fluids has been formulated by Stokes [6]. One of the applications of couple-stress fluids is the study of the mechanisms of lubrication of synovial joints. A human joint is a dynamically loaded bearing which has an articular cartilage as bearing and a synovial fluid as lubricant. When a fluid film is generated, squeeze-film action is capable of providing considerable protection to the cartilage surface. The shoulder, ankle, knee, and hip joints are loaded bearing synovial joints of the human body and these joints have a low friction coefficient and negligible wear. The normal synovial fluid is a viscous, non-Newtonian fluid and is generally clear or yellowish.

The theory due to Stokes [6] allows for polar effects such as the presence of couple stresses and body couples. Stokes [6] theory has been applied to the study of some simple lubrication problems (see e. g. Sinha et al. [9]; Bujurke and Jayaraman [10]; Lin [11]). According to the theory of Stokes [6], couple stresses are found to appear in noticeable magnitudes in fluids with very large molecules. Since the long-chain hyaluronic acid molecules are found as additives in synovial fluid, Walicki and Walicka [12] modelled the synovial fluid as couple-stress fluid in human joints. The synovial fluid is the natural lubricant of joints of the vertebrates. The detailed description of the joint lubrication has very important practical applications. Practically all diseases of joints are caused by or connected with a malfunction of the lubrication. The efficiency of the physiological joint lubrication is caused by several mechanisms. The synovial fluid is, due to its content of the hyaluronic acid, a fluid of high viscosity, near to a gel. Goel et al. [13] have studied the hydromagnetic stability of an unbounded couple-stress binary fluid mixture under rotation with vertical temperature and concentration gradients. Sharma et al. [14] have considered a couple-stress fluid with suspended particles heated from below. Kumar et al. [15] have considered the thermal instability of a layer of couple-stress fluid acted on by a uniform rotation and found that, for stationary convection, the rotation has a stabilizing effect, whereas the couple-stress has both stabilizing and destabilizing effects. An electrically conducting couple-stress fluid heated from below in porous medium in presence of a magnetic field and rotation have been studied by Sunil et al. [16]. The use of magnetic field is being made for the clinical purposes in detection and cure of certain diseases with the help of magnetic field devices/instruments.

Keeping in mind the importance of non-Newtonian fluids and convection in fluid heated from below, the present paper is devoted to study the compressible couple-stress fluid heated from below in presence of a magnetic field.

2. Formulation of the Problem and Perturbation Equations

Here we consider an infinite, horizontal, compressible, electrically conducting couple-stress fluid layer of thickness d, heated from below so that a uniform temperature gradient $\beta = |\mathrm{d}T/\mathrm{d}z|$ is maintained. This layer is acted on by an uniform vertical magnetic field $\vec{H}(0,0,H)$ and the gravity field $\vec{g}(0,0,-g)$.

The initial state is, therefore, a state in which the fluid velocity, temperature, pressure, and density at any point in the fluid are given by

$$\vec{v} = 0, \quad T = T(z), \quad p = p(z), \quad \rho = \rho(z), \tag{1}$$

respectively, where

$$T(z) = T_0 - \beta z,$$

$$p(z) = p_\mathrm{m} - g\int_0^z (\rho_\mathrm{m} + \rho_0)\mathrm{d}z,$$

$$\rho(z) = \rho_\mathrm{m}[1 - \alpha_\mathrm{m}(T - T_\mathrm{m}) + K_\mathrm{m}(p - p_\mathrm{m})],$$

and

$$\alpha_\mathrm{m} = -\left(\frac{1}{\rho}\frac{\partial\rho}{\partial T}\right)_\mathrm{m} \quad (= \alpha, \text{ say}), \tag{2}$$

$$K_\mathrm{m} = \left(\frac{1}{\rho}\frac{\partial\rho}{\partial p}\right)_\mathrm{m}. \tag{3}$$

Let $\vec{q}(u,v,w)$, θ, δp, $\delta\rho$, and $\vec{h}(h_x,h_y,h_z)$ denote, respectively, the perturbations in velocity $(0,0,0)$, temperature T, pressure p, density ρ, and magnetic field $\vec{H}(0,0,H)$. Then the linearized perturbation equations relevant to the problem (Stokes [6], Sharma [5]) are

$$\frac{\partial\vec{q}}{\partial t} = -\frac{1}{\rho_\mathrm{m}}\nabla\delta p - g\alpha\theta + \left(\nu - \frac{\mu}{\rho_\mathrm{m}}\nabla^2\right)\nabla^2\vec{q} + \frac{\mu_\mathrm{e}}{4\pi\rho_\mathrm{m}}(\nabla\times\vec{h})\times\vec{H}, \tag{4}$$

$$\nabla\cdot\vec{q} = 0, \tag{5}$$

$$\nabla\cdot\vec{h} = 0, \tag{6}$$

$$\frac{\partial\vec{h}}{\partial t} = (\vec{H}\cdot\nabla)\vec{q} + \eta\nabla^2\vec{h}, \tag{7}$$

$$\frac{\partial\theta}{\partial t} = \left(\beta - \frac{g}{c_\mathrm{p}}\right)w + \kappa\nabla^2\theta. \tag{8}$$

Here ν, μ', c_p, and κ stand for kinematic viscosity, couple-stress viscosity, specific heat at constant pressure, and thermal diffusivity, respectively.

The equation of state is

$$\rho = \rho_\mathrm{m}[1 - \alpha(T - T_0)],$$

where α is the coefficient of thermal expansion. Therefore, the change in density $\delta\rho$ caused by the perturbation θ in temperature is given by

$$\delta\rho = -\rho_\mathrm{m}\alpha\theta.$$

Equations (4) – (8) give

$$\frac{\partial}{\partial t}\cdot(\nabla^2 w) - \left(\nu - \frac{\mu}{\rho_\mathrm{m}}\nabla^2\right)\nabla^4 w - g\alpha\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\theta - \frac{\mu_\mathrm{e}}{4\pi\rho_\mathrm{m}}\frac{\partial}{\partial z}\nabla^2 h_z = 0, \tag{9}$$

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right)h_z = H\frac{\partial w}{\partial z}, \tag{10}$$

$$\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)\theta = \left(\beta - \frac{g}{c_\mathrm{p}}\right)w, \tag{11}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

3. The Dispersion Relation

We now analyze the disturbances into normal modes, assuming that the perturbation quantities have the space and time dependence of the form

$$[w, \theta, h_z] = [W(z), \Theta(z), K(z)] \exp(\mathrm{i}k_x x + \mathrm{i}k_y y + nt), \quad (12)$$

where k_x, k_y are wave numbers along the x- and y-direction respectively, $k = \sqrt{k_x^2 + k_y^2}$ is the resultant wave number, and n is, in general, a complex constant. Using expression (12), equations (9) – (11), in non-dimensional form, become

$$\begin{aligned}\sigma(D^2 - a^2)W &+ \frac{g\alpha d^2 a^2}{\nu}\Theta \\ &= [1 - F(D^2 - a^2)][(D^2 - a^2)^2 W] \\ &\quad + \frac{\mu_e H d}{4\pi\rho_m \nu}(D^2 - a^2)DK,\end{aligned} \quad (13)$$

$$(D^2 - a^2 - p_2\sigma)K = -\frac{Hd}{\eta}DW, \quad (14)$$

$$(D^2 - a^2 - p_1\sigma)\Theta = -\left(\frac{G-1}{G}\right)\frac{\beta d^2}{\kappa}W. \quad (15)$$

Here we have put $a = kd$, $\sigma = nd^2/\nu$, $x/d = x^*$, $y/d = y^*$, $z/d = z^*$, and $D = \mathrm{d}/\mathrm{d}z^*$. Here $p_1 = \nu/\kappa$ is the Prandtl number, $F = \mu^*/\rho_0 d^2 \nu$ is the dimensionless couple-stress parameter, $G = c_p\beta/g$ is the dimensionless compressibility parameter.

Consider the case in which both the boundaries are free and are maintained at constant temperatures. The boundary conditions appropriate to the problem (Chandrasekhar [4]) are

$$W = 0, \quad \Theta = 0, \quad DK = 0, \quad \text{at } z = 0 \text{ and } z = 1. \quad (16)$$

The constitutive equations for the couple-stress fluid are

$$\tau_{ij} = (2\mu - 2\mu^* \nabla^2)e_{ij}, \quad e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \quad (17)$$

The conditions on a free surface are vanishing of tangential stresses τ_{xz} and τ_{yz}, which yield

$$\tau_{xz} = (\mu - \mu^* \nabla^2)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0, \quad (18)$$

$$\tau_{yz} = (\mu - \mu^* \nabla^2)\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0. \quad (19)$$

Since w vanishes for all x and y on the bounding surface, it follows from (18) and (19) that

$$(\mu - \mu^* \nabla^2)\frac{\partial u}{\partial z} = 0, \quad (\mu - \mu^* \nabla^2)\frac{\partial v}{\partial z} = 0. \quad (20)$$

From the equation of continuity (5), differentiated with respect to z, we conclude that

$$\left[\mu - \mu^*\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right]\frac{\partial^2 w}{\partial z^2} = 0, \quad (21)$$

which, by using (12) and (16), implies that

$$D^2 W = 0, \quad D^4 W = 0 \quad \text{at } z = 0 \text{ and } z = 1. \quad (22)$$

Using the boundary conditions (16) and (22), it can be shown with the help of (13) – (15) that all the even order derivatives of W must vanish at $z = 0$ and $z = 1$. Hence, the proper solution of W characterizing the lowest mode is

$$W = W_0 \sin \pi z, \quad (23)$$

where W_0 is a constant.

Eliminating Θ, K between (13) – (15), we get

$$\begin{aligned}&\sigma\{1 - F(D^2 - a^2)\}[D^2 - a^2]^2[D^2 - a^2 - p_1\sigma] \\ &\cdot [D^2 - a^2 - p_2\sigma]W + Ra^2\frac{G-1}{G}(D^2 - a^2 - p_2\sigma)W \\ &- Q(D^2 - a^2)(D^2 - a^2 - p_1\sigma)D^2 W = 0,\end{aligned} \quad (24)$$

where $R = g\alpha\beta d^4/\nu\kappa$ is the Rayleigh number and $Q = \mu_e H^2 d^2/4\pi\rho_m \nu\eta$ is the Chandrasekhar number.

Using (23), (24) yields

$$\begin{aligned}R_1 = \frac{G}{G-1}&(1+x)(1+x+\mathrm{i}p_1\sigma_1)\Big[1+x \\ &+ \mathrm{i}p_2\sigma_1\{\mathrm{i}\sigma_1 + (1+F_1\overline{1+x})(1+x)\} + Q_1\Big] \\ &\cdot [x(1+x+\mathrm{i}p_2\sigma_1)]^{-1},\end{aligned} \quad (25)$$

where $R_1 = R/\pi^4$, $Q_1 = Q/\pi^2$, $a^2 = \pi^2 x$, $\mathrm{i}\sigma_1 = \sigma/\pi^2$ and $F_1 = \pi^2 F$.

4. The Stationary Convection

For stationary convection, $\sigma = 0$ and (25) reduces to

$$R_1 = \left(\frac{G}{G-1}\right)\frac{1+x}{x}[(1+x)^2 + F_1(1+x)^3 + Q_1]. \quad (26)$$

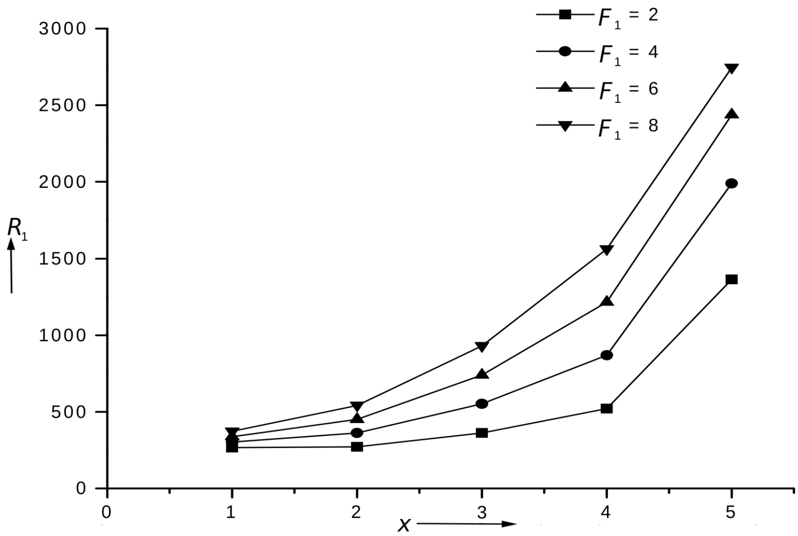

Fig. 1. Variation of Rayleigh number R_1 with wave number x for $Q_1 = 100$ and $G = 10$.

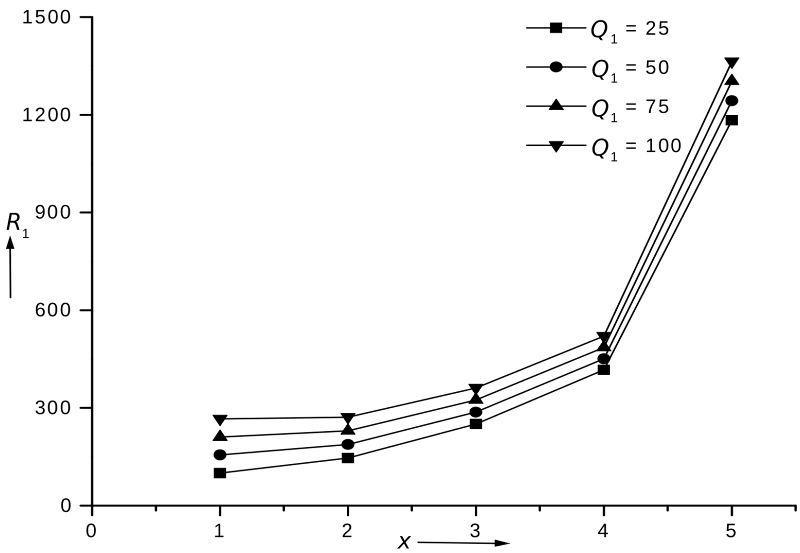

Fig. 2. Variation of Rayleigh number R_1 with wave number x for $F_1 = 2$ and $G = 10$.

(26) yields

$$\frac{dR_1}{dF_1} = \frac{G}{G-1}\frac{(1+x)^4}{x}, \tag{27}$$

$$\frac{dR_1}{dQ_1} = \frac{G}{G-1}\frac{1+x}{x}, \tag{28}$$

which are always positive. The couple-stress and magnetic field, thus, have stabilizing effects on the thermal instability of couple-stress fluid, for the stationary convection.

For fixed Q_1 and F_1, let G (accounting for the compressibility effects) be also kept fixed in (26). Then we find that

$$\overline{R_c} = \left(\frac{G}{G-1}\right) R_c, \tag{29}$$

where $\overline{R_c}$ and R_c denote, respectively, the critical Rayleigh numbers in the presence and absence of compressibility. The effect of compressibility is thus to postpone the onset of thermal convection when $G > 1$. The compressibility, therefore, has a stabilizing effect on the thermal convection in the couple-stress fluid in hydromagnetics.

The results have been shown graphically also. In Figure 1, the variation of Rayleigh number R_1 with wave number x, for $Q_1 = 100$ and $G = 10$ is considered when the couple-stress parameter F_1 is varied. It is clear from the graph that Rayleigh number R_1 increases with the increase in the value of F_1 thus implying stabilizing effect of the couple-stress parameter. In Figure 2, the variation of Rayleigh number R_1 with wave number x for $F_1 = 2$ and $G = 10$ is shown, whereas Chandrasekhar number Q_1 is varied. It is clear from the graph that with the increase in the value of Q_1, there is an increase in the value of Rayleigh number R_1 which suggests that the presence of a magnetic field causes the stabilizing effect for $G > 1$.

5. Some Important Theorems

Theorem 1: The system is stable for $G < 1$.

Proof: Multiplying (9) by W^*, the complex conjugate of W, and using (10), (11), and the boundary conditions (16) and (22), we obtain

$$\sigma I_1 - \frac{g\alpha a^2\kappa}{\nu\beta}\left(\frac{G}{G-1}\right)(I_2 + p_1\sigma^* I_3) + (I_6 + FI_7) + \frac{\mu_e\eta}{4\pi\rho_m\nu}[I_4 + \sigma^* p_2 I_5] = 0, \tag{30}$$

where

$$\begin{aligned} I_1 &= \int_0^1 (|DW|^2 + a^2|W|^2)\mathrm{d}z, \\ I_2 &= \int_0^1 (|D\Theta|^2 + a^2|\Theta|^2)\mathrm{d}z, \\ I_3 &= \int_0^1 |\Theta|^2\mathrm{d}z, \\ I_4 &= \int_0^1 (|D^2K|^2 + 2a^2|DK|^2 + a^4|K|^2)\mathrm{d}z, \\ I_5 &= \int_0^1 (|DK|^2 + a^2|K|^2)\mathrm{d}z, \\ I_6 &= \int_0^1 (|D^2W|^2 + 2a^2|DW|^2 + a^4|W|^2)\mathrm{d}z, \\ I_7 &= \int_0^1 (|D^3W|^2 + a^6|W|^2 + 3a^2|D^2W|^2 + 3a^4|DW|^2)\mathrm{d}z, \end{aligned} \tag{31}$$

where σ^* is the complex conjugate of σ. The integrals $I_1 - I_7$ are all positive definite.

Putting $\sigma = \sigma_r + i\sigma_i$ in (30) and equating real and imaginary parts, we obtain

$$\sigma_r\left[I_1 - \frac{g\alpha a^2\kappa}{\nu\beta}p_1 I_3\frac{G}{G-1} + \frac{\eta\mu_e}{4\pi\rho_m\nu}p_2 I_5\right] = -\left[-\frac{g\alpha a^2}{\nu\beta}\left(\frac{G}{G-1}\right)\kappa I_2 + \frac{\mu_e\eta}{4\pi\rho_m\nu}I_4 + I_6 + FI_7\right], \tag{32}$$

and

$$\sigma_i\left[I_1 + \frac{g\alpha a^2\kappa}{\nu\beta}\left(\frac{G}{G-1}\right)p_1 I_3 - \frac{\eta\mu_e}{4\pi\rho_m\nu}p_2 I_5\right] = 0. \tag{33}$$

It is evident from (32) that if $G < 1$, σ_r is negative meaning thereby the stability of the system.

Theorem 2: The modes may be oscillatory or non-oscillatory in contrast to the case of no magnetic field, where modes are non-oscillatory for $G > 1$.

Proof: (33) yields that $\sigma_i = 0$ or $\sigma_i \neq 0$ which means that modes may be non-oscillatory or oscillatory. In the absence of a magnetic field, (33) gives

$$\sigma_i\left[I_1 + \frac{g\alpha a^2\kappa}{\nu\beta}\frac{G}{G-1}p_1 I_3\right] = 0 \tag{34}$$

and the terms in brackets are positive definite when $G > 1$. Thus $\sigma_i = 0$, which means that oscillatory modes are not allowed and the principle of exchange of stabilities is satisfied in the absence of a magnetic field. This result is true for compressible, couple-stress fluids as well as for incompressible Newtonian fluids (Chandrasekhar [4]) in the absence of a magnetic field. The presence of a magnetic field brings oscillatory modes (as σ_i may not be zero) which were non-existent in its absence. (32) simply tells that there may be stability or instability in the presence of a magnetic field in compressible couple-stress fluids, which is also true in the absence of a magnetic field as well as in incompressible, Newtonian fluids (Chandrasekhar [4]).

Theorem 3: $\kappa < \eta$ is the sufficient condition for the non-existence of overstability.

Proof: For overstability, we put $\frac{\sigma}{\pi^2} = i\sigma_1$ where σ_1 is real. Then (25) can be written as

$$R_1 x(1+x+ip_2\sigma_1) = \frac{G}{G-1}(1+x)(1+x+ip_1\sigma_1)\Big[(1+x+ip_2\sigma_1) \cdot \left\{i\sigma_1 + (1+F_1\overline{1+x})(1+x)\right\} + Q_1\Big]. \tag{35}$$

Since for overstability we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations. It is suffice to find conditions for which (35) will admit of solutions with σ_1 real. Equating real and imaginary parts of (35) and eliminating R_1 between them, we obtain

$$p_1 p_2^2 \alpha \sigma_1^2 + [p_2 \alpha^3 (p_1 - p_2) + p_2 F_1 \alpha (p_1 - p_2)]\sigma_1 + [\alpha^4 (p_1 - p_2) + \alpha^5 F_1 (p_1 - p_2) + Q_1 \alpha (p_1 - p_2)] = 0, \tag{36}$$

which is of the form

$$A\sigma_1^2 + B\sigma_1 + C = 0,$$

where $1 + x = \alpha$,

$$A = p_1 p_2^2 \alpha,$$

$$B = (-\alpha p_2 - p_2^2 \alpha^3 - p_2^2 \alpha^4 F_1 + p_2 \alpha^3 + p_1 p_2 \alpha^3 + p_1 p_2 F_1 \alpha) = p_2 \alpha^3 (p_1 - p_2) + p_2 F_1 \alpha (p_1 - p_2),$$

$$C = \big(- \alpha^4 p_2 - F_1 \alpha^5 p_2 - Q \alpha^2 p_2 + p_1 \alpha^3 + p_1 \alpha^4 + p_1 \alpha^5 F_1 + p_1 Q_1 \alpha \big) = \alpha^4 (p_1 - p_2) + \alpha^5 F_1 (p_1 - p_2) + Q_1 \alpha (p_1 - p_2).$$

(36) is quadratic in σ_1, as σ_1 is real for overstability, ${\sigma^2}_1$ is positive. It is evident from (36) that $p_1 > p_2$, thus implying

$$\frac{\nu}{\kappa} > \frac{\nu}{\eta},$$

i. e. $\kappa < \eta$ is, therefore, a sufficient condition for the non-existence of overstability, the violation of which does not necessarily imply the occurrence of overstability. The sufficient condition for the non-existence of overstability is found to be the same for compressible, couple-stress fluids as well as for incompressible Newtonian fluids (Chandrasekhar [4]), in the presence of a magnetic field, which is heated from below.

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