On the Reconstruction of the First Term in the Variational Iteration Method for Solving Differential Equations

Mehdi Tatari[a] and Mehdi Dehghan[b]

[a] Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, 84156-83111, Iran
[b] Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, No. 424, Hafez Ave., Tehran, Iran

Reprint requests to M. D.; E-mail: mdehghan@aut.ac.ir

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The well-known He's variational iteration method is an efficient and easy technique for finding the solution of various kinds of problems. In this method the solution of a nonlinear problem is found without any need to solve a linear or nonlinear system of equations. In the current work we present an idea for accelerating the convergence of the resulted sequence to the solution of the problem by choosing a suitable initial term. The efficiency of the method will be shown by testing the new algorithm on several examples.

Key words: Variational Iteration Method; Initial Term; Differential Equations; Partial Differential Equations; Integral Equations.

1. Introduction

In the recent years, the variational iteration method (VIM) has been used by researchers in science and engineering for studying problems which appear in modelling of various phenomena. Most models which describe the behaviour of a physical problem are nonlinear. Having nonlinear terms in the mathematical models usually causes instabilities in the numerical methods. In fact, small perturbations in the given data make inaccurate approximation for the solution. Also in most problems, it is impossible to find the solution by analytical techniques.

According to the above discussion, He's variational iteration method has been used for solving different kinds of problems in various field of sciences and engineering. Some important references in this topic are [1 – 3]. The variational iteration method has been employed in [4] for solving some models in biology. Also this method is used for solving a wave equation subject to an integral conservation condition in [5]. A complete discussion of the method is presented in [6, 7]. The application of the present method for solving integral equations has been investigated in [8]. This method has been implemented for solving problems in calculus of variations in [9]. As an application of this method for solving problems in astrophysics, we refer to [10]. He's variational iteration method is used in [11] to solve two systems of Volterra integro-differential equations. This technique is employed in [12] to find the numerical solutions of the Korteweg-de Vries and the modified Korteweg-de Vries equations with given initial conditions. Also this method is compared with the Adomian decomposition method [13 – 16], and the efficiency of the new approach is seen in the test problems. Also author of [17] investigated solutions of nonlinear dispersive equations $k(2,2,1)$ and $k(3,3,1)$ by using this technique. It is used in [18] to find the solution of the Klein-Gordon partial differential equation. The convergence of He's variational iteration method is investigated in [19]. To see more information about the variational iteration method and its applications see [20 – 30].

As a very important point in He's variational iteration method, the choice of the initial guess can be investigated. This choice of initial guess is important especially in the cases when finding more iterations is impossible or needs a large amount of time. The emphasis of this paper is on finding a better choice of the initial term in the variational iteration method.

The organization of this paper is as follows:

In Section 2, a new method has been introduced for finding a better choice of the initial guess in He's variational iteration method. Some examples are given in

Section 3 to show the efficiency of the new method presented in this work. A conclusion has been drawn in Section 4.

2. The Method of Solution

In this section we construct a suitable initial term in the variational iteration method for solving differential equations. At first we consider the following initial value problem:

$$y'' = f(t, y, y'), \tag{1}$$

where $y(t_0)$ and $y'(t_0)$ are given, and we assume that f satisfies the conditions which guarantee the existence and uniqueness of the problem. The variational iteration method finds the solution with an initial guess for y_0 and finds other components with the following recurrence formula:

$$y_{n+1} = y_n + \int_0^s \lambda(y_n''(s) - f(s, y_n(s), y_n'(s)))\mathrm{d}s.$$

The choice of y_0 has an important role in the convergence behaviour. Usually, the initial term is selected as $y_0 = y(t_0) + ty'(t_0)$. A better choice of initial term is

$$\tilde{y}_0 = y(t_0) + (t - t_0)y'(t_0) + \frac{(t - t_0)^2}{2!}y''(t_0),$$

where $y''(t_0) = f(t_0, y(t_0), y'(t_0))$ using (1). In fact this choice of initial term produces the first three terms of the first partial summation of the Taylor expansion of y. Partial summations with more terms can be used, but finding higher derivatives of y from (1) may be very complicated. Note that this idea can be implemented for solving higher-order ordinary differential equations. For solving the partial differential equations in the form

$$u_{tt} = L(u),$$

with the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, a similar technique can be applied to find a more suitable initial guess. In this case the initial term is considered as

$$\tilde{u}_0 = f(x) + tg(x) + \frac{t^2}{2!}(u_1)_{tt}(x,0),$$

where u_1 is the first iteration of He's variational iteration method which is found by the initial term $u_0 = f(x) + tg(x)$. The choice of the initial guess is important especially, when finding the iterations is impossible or difficult.

As an important point we should note that in some cases He's variational iteration method converges to Taylor's expansion of the method in the t direction. More precisely, if u_n be the n-th iteration in He's method we have

$$u_n = \sum_{k=0}^{n} \frac{t^k}{k!}\frac{\partial^k u}{\partial t^k}(x,0).$$

Therefore, we obtain

$$u_1 = u(x,0) + tu_t(x,0) + \frac{t^2}{2}u_{tt}(x,0),$$

so

$$(u_1)_{tt}(x,0) = u_{tt}(x,0).$$

In this case we can write

$$\tilde{u}_0 = u(x,0) + tu_t(x,0) + \frac{t^2}{2}u_{tt}(x,0),$$

i. e.

$$\tilde{u}_0 = u_1,$$

which results in

$$\tilde{u}_n = u_{n+1}$$

for $n \geq 1$.

In this case, an efficient aftertreatment technique has been proposed for extending the domain of the convergence [31]. In this procedure the Padé approximation has been used to modify the approximate solution of the problem. In the next section we examine the new idea presented in the current paper and apply it to several examples.

3. Test Examples

3.1. Example 1

Consider Duffing's equation [32]

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 3y - 2y^3 = \cos t \sin 2t,$$

with the initial conditions $y(0) = 0$, and $y'(0) = 1$. The analytical solution of this equation is $y = \sin t$. He's variational iteration method for this problem generates the sequence [6]

$$y_{n+1} = y_n + \frac{1}{\sqrt{3}}\int_0^t \sin(\sqrt{3}(s-t)) \cdot (y_n'' + 3y_n - 2y_n^3 - \cos s \sin 2s)\mathrm{d}s.$$

Table 1. Relative errors $y_1(0.1i)$ and $\tilde{y}_1(0.1i)$ for $i = 1,\ldots,10$.

i	$\mathrm{Rel}(y_1(0.1i))$	$\mathrm{Rel}(\tilde{y}_1(0.1i))$
1	0.23859×10^{-7}	0.39644×10^{-8}
2	0.15232×10^{-5}	0.12660×10^{-7}
3	0.17335×10^{-4}	0.56717×10^{-7}
4	0.97518×10^{-4}	0.47429×10^{-6}
5	0.37204×10^{-3}	0.26725×10^{-5}
6	0.11112×10^{-2}	0.11274×10^{-4}
7	0.28036×10^{-2}	0.38188×10^{-4}
8	0.62529×10^{-2}	0.10953×10^{-3}
9	0.12694×10^{-1}	0.27638×10^{-3}
10	0.23936×10^{-1}	0.62990×10^{-3}

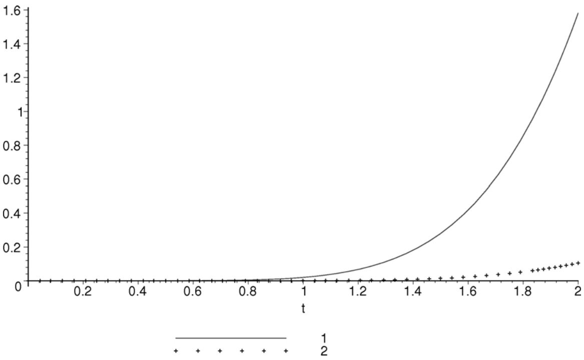

Fig. 1. Plot of the functions $|y(t)-y_1(t)|$ and $|y(t)-\tilde{y}_1(t)|$ (2).

This problem has been solved using the initial guess $y_0 = t$. For finding the better approximation, the values of $y''(0)$ and $y^3(0)$ are founded and the first term is considered as

$$\tilde{y}_0 = y_0 + \frac{t^2}{2!}y''(0) + \frac{t^3}{3!}y^3(0).$$

In Figure 1, the errors $|y(t)-y_1(t)|$ and $|y(t)-\tilde{y}_1(t)|$ have been plotted. In this problem finding more iterations needs complicated computations. Therefore, finding a suitable initial guess is important. In Table 1, the values of the relative errors of $y_1(t)$ and $\tilde{y}_1(t)$ have been presented for some values of t.

3.2. Example 2

To show the efficiency of the idea presented in the previous section, in this example we consider the following integro-differential equation [8]:

$$u''(x) = 1 + x\exp(x) - \int_0^x \exp(x-t)u(t)\mathrm{d}t, \quad (2)$$

with the conditions $u(0) = 0$ and $u'(0) = 1$. The exact solution of this problem is $u(x) = \exp(x) - 1$. He's variational iteration method for solving this problem is written as

$$u_{n+1} = u_n + \int_0^x (s-x)(u''(s) - 1 - s\exp(s) + \int_0^s \exp(s-t)u(t)\mathrm{d}t)\mathrm{d}s.$$

Table 2. Values of $|u(0.1i) - u_2(0.1i)|$ and $|u(0.1i) - \tilde{u}_2(0.1i)\|$ for $i = i,\ldots,10$.

i	$\lvert u(0.1i)-u_2(0.1i)\rvert$	$\lvert u(0.1i)-\tilde{u}_2(0.1i)\rVert$
1	0.25645×10^{-12}	0.28403×10^{-14}
2	0.67899×10^{-10}	0.14987×10^{-11}
3	0.18001×10^{-8}	0.59400×10^{-10}
4	0.18605×10^{-7}	0.81575×10^{-9}
5	0.11477×10^{-6}	0.62684×10^{-8}
6	0.51086×10^{-6}	0.33364×10^{-7}
7	0.18155×10^{-5}	0.13784×10^{-6}
8	0.54720×10^{-5}	0.47312×10^{-6}
9	0.14544×10^{-4}	0.14096×10^{-5}
10	0.35007×10^{-4}	0.37563×10^{-5}

In Table 2, the results of the variational iteration method have been presented by the following initial terms $u_0 = x$, $\tilde{u}_0 = x + \frac{u''(0)}{2}x^2$, where $u''(0)$ can be found using (2).

3.3. Example 3

In this example we consider the classical wave equation as

$$u_{tt} = u_{xx},$$

with the initial conditions $u(x,0) = \frac{1}{8}\sin(\pi x)$ and $u_t(x,0) = 0$. The exact solution of this problem is $u(x,t) = \frac{1}{8}\sin(\pi x)\cos(\pi t)$. By the initial term $u_0 = \frac{1}{8}\sin(\pi x)$, He's variational iteration method is written as

$$u_{n+1} = u_n + \int_0^t (s-t)((u_n)_{tt} - (u_n)_{xx})\mathrm{d}s,$$

where $(u_n)_{tt} = \frac{\partial^2 u_n}{\partial t^2}$ and $(u_n)_{xx} = \frac{\partial^2 u_n}{\partial x^2}$. In this example the variational iteration method provides the partial summation of the Taylor series. So, as we said in Section 2, by choosing the initial guesses

$$u_0 = \frac{1}{8}\sin(\pi x)$$

and

$$\tilde{u}_0 = u_0 + \frac{t^2}{2}(u_1)_{tt}(x,0)$$

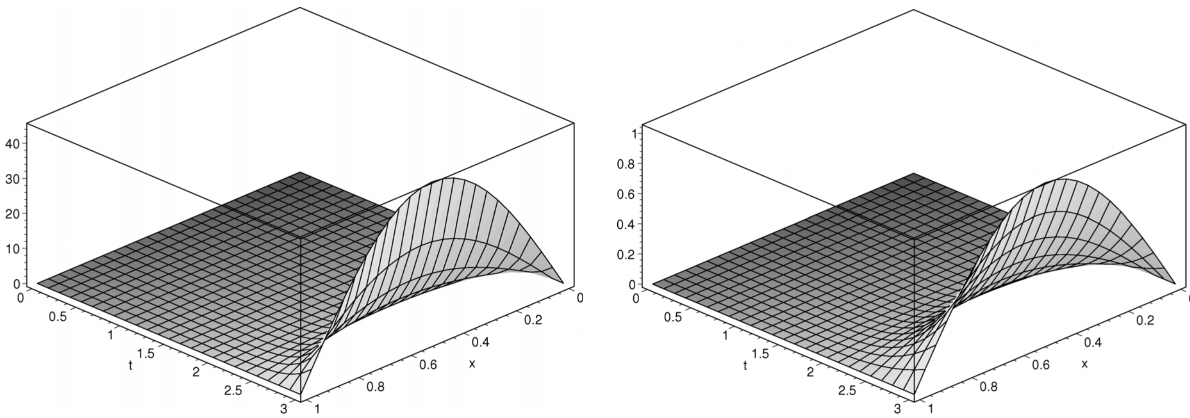

Fig. 2. Plot of the functions $|u(x,t)-u_6(x,t)|$ (left) and $|u(x,t)-[\frac{6}{6}](x,t)\|$ (right).

Table 3. Values of $|u(0.5,0.5i)-u_6(0.5,0.5i)|$ and $|u(0.5,0.5i)-[\frac{6}{6}](0.5,0.5i)|$ for $i=1,\dots,10$.

i	$\lvert u(0.5,0.5i)-u_6(0.5,0.5i)\rvert$	$\lvert u(0.5,0.5i)-[\frac{6}{6}](0.5,0.5i)\rvert$
1	0.79018×10^{-9}	0.82085×10^{-9}
2	0.12558×10^{-4}	0.91885×10^{-5}
3	0.34892×10^{-2}	0.14301×10^{-2}
4	0.18316	0.34044×10^{-1}
5	0.38366×10^{1}	0.27020
6	0.44815×10^{2}	0.10359×10^{1}
7	0.39448×10^{3}	0.23804×10^{1}
8	0.20282×10^{4}	0.38233×10^{1}
9	0.94029×10^{4}	0.49653×10^{1}
10	0.36568×10^{5}	0.59860×10^{1}

Table 4. Relative errors $u_2(1,0.1i)$ and $\tilde{u}_2(1,0.1i)$ for $i=1,...,10$.

i	$\mathrm{Rel}(u_2(1,0.1i))$	$\mathrm{Rel}(\tilde{u}_2(1,0.1i))$
1	0.14943×10^{-10}	0.30915×10^{-14}
2	0.38741×10^{-8}	0.31924×10^{-11}
3	0.10143×10^{-6}	0.18647×10^{-9}
4	0.10447×10^{-5}	0.33738×10^{-8}
5	0.64866×10^{-5}	0.32223×10^{-7}
6	0.29381×10^{-4}	0.20616×10^{-6}
7	0.10762×10^{-3}	0.10042×10^{-5}
8	0.33943×10^{-3}	0.40254×10^{-5}
9	0.96182×10^{-3}	0.13986×10^{-4}
10	0.25281×10^{-2}	0.43763×10^{-4}

we have

$$\tilde{u}_n = u_{n+1}.$$

In Figure 2, the functions $|u(x,t)-u_6(x,t)|$ and $|u(x,t)-[\frac{6}{6}](x,t)|$ are plotted for some values of x and t, where $[\frac{6}{6}](x,t)$ is the Padé approximation of the partial summation $u_6(x,t)$. Note that in this case $u_6(x,t)$ is a polynomial of degree 12 in t. In Table 3, some values of $|u(x,t)-u_6(x,t)|$ and $|u(x,t)-[\frac{6}{6}](x,t)|$ are shown.

3.4. Example 4

In this example the Klein-Gordon equation is investigated. As the first case we consider the nonlinear Klein-Gordon equation with quadratic nonlinearity [20]

$$u_{tt}+\alpha u_{xx}+\beta u+\gamma u^2 = f(x,t),$$

with $\alpha=-1$, $\beta=0$, and $\gamma=1$. The initial conditions are given by $u(x,0)=x$ and $u_t(x,0)=0$ the right hand side function is

$$f(x,t) = -x\cos t + x^2\cos^2 t.$$

The analytical solution is given as $u(x,t)=x\cos(t)$. Using He's variational iteration method [6] we get

$$u_{n+1} = u_n + \int_0^t (s-t)((u_n)_{tt}-(u_n)_{xx}+u_n^2 + x\cos s - x^2\cos^2 s)\mathrm{d}s,$$

with the initial term $u_0 = x$, the problem has been solved. Also based on the first iteration of the sequence, a new improved initial term has been constructed. The results of the two experiments have been shown in Figure 3. In this problem finding more iterations of He's variational iteration method is very time consuming, so modification of the initial guess can provide an accurate and fast approximation using a few terms of iterations. In Table 4, the relative errors of $u_2(1,0.1i)$ and $\tilde{u}_2(1,0.1i)$ have been presented for $i=1,\dots,10$. From the given data, the effect of the reconstruction of the initial guess is clear.

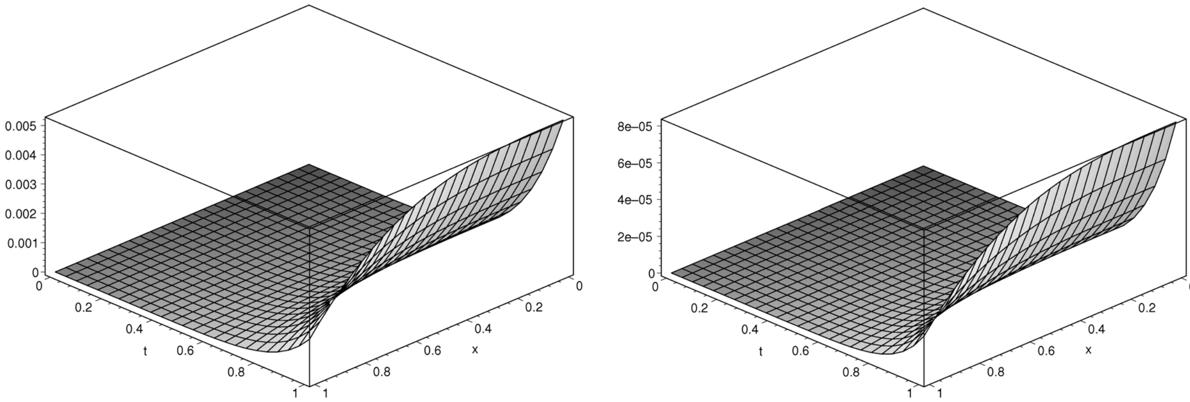

Fig. 3. Plot of the functions $|u(x,t)-u_2(x,t)|$ (left) and $|u(x,t)-\tilde{u}_2(x,t)|$ (right).

Table 5. Relative errors $u_1(1,0.1i)$ and $\tilde{u}_1(1,0.1i)$ for $i = 1,\ldots,10$.

i	$\mathrm{Rel}(u_1(1,0.1i))$	$\mathrm{Rel}(\tilde{u}_1(1,0.1i))$
1	0.68457×10^{-5}	0.16722×10^{-7}
2	0.14494×10^{-3}	0.12195×10^{-5}
3	0.90725×10^{-3}	0.22790×10^{-4}
4	0.33965×10^{-2}	0.15368×10^{-3}
5	0.95385×10^{-2}	0.64645×10^{-3}
6	0.22274×10^{-1}	0.20533×10^{-2}
7	0.45743×10^{-1}	0.54066×10^{-2}
8	0.85468×10^{-1}	0.12445×10^{-1}
9	0.14855	0.28893×10^{-1}
10	0.24394	0.49794×10^{-1}

In the next part of this example we consider the Klein-Gordon equation with cubic nonlinearity [20] $u_{tt}-u_{xx}+u+u^3=f(x,t)$, with the right hand side

$$f(x,t)=(x^2-2)\cosh(x+t)-4x\sinh(x+t) + x^6\cosh^3(x+t)$$

and the initial conditions

$$u(x,0)=x^2\cosh(x),$$
$$u_t(x,0)=x^2\sinh(x).$$

The analytical solution of this problem is $u(x,t)=x^2\cosh(x+t)$. In this example, He's variational iteration method has been implemented for solving this problem with $u_0=x^2\cosh(x)+tx^2\sinh(x)$. It is possible to find only the first iteration of the sequence as follows [6]:

$$u_1=u_0-\int_0^t \sin(s-t) \cdot((u_0)_{tt}-(u_0)_{xx}+u_0+u_0^3-f(x,s))\mathrm{d}s.$$

Finding other terms need to compute integrals which can not be evaluated analytically. By the technique presented in this report a new first term is founded as

$$\tilde{u}_0=x^2\cosh(x)+tx^2\sinh(x)+\frac{t^2}{2!}(u_1)_{tt}.$$

In Table 5, the relative errors of $u_1(1,0.1i)$ and $\tilde{u}_1(1,0.1i)$ have been presented for $i=1,\ldots,10$. By employing the new procedure we found that the accuracy of the approximate solution with only one iteration is more than the standard case.

4. Conclusion

Finding the initial term in most iterative methods is very important. In the present work the choice of the initial terms in the variational iteration method has been investigated. The obtained results for ordinary, partial differential, and integro-differential equations show the efficiency of the method by choosing the described initial terms. This idea is efficient especially when finding the iterations of He's variational iteration method is impossible or difficult.

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