New Soliton Solutions of Chaffee-Infante Equations Using the Exp-Function Method

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Z. Naturforsch. **65a,** 197 – 202 (2010); received November 12, 2008 / revised July 28, 2009

In this paper, the exp-function method is applied by using symbolic computation to construct a variety of new generalized solitonary solutions for the Chaffee-Infante equation with distinct physical structures. The results reveal that the exp-function method is suited for finding travelling wave solutions of nonlinear partial differential equations arising in mathematical physics.

Key words: Chaffee-Infante Equations; Solitonary Solutions; Travelling Wave Solutions; Exp-Function Method.

PACS numbers: 02.30.Jr; 04.20.Jb

1. Introduction

Large varieties of physical, chemical, and biological phenomena are governed by nonlinear partial differential equations. Solving nonlinear equations may guide authors to know the described process deeply and sometimes leads them to know some facts that are not simply understood through common observations. The investigation of exact solutions of nonlinear partial differential equations plays an important role in mathematics and physics. A variety of powerful methods have been developed for obtaining approximate and exact solutions for various nonlinear equations like sine-cosine method [1], Adomian decomposition method [2], variational iteration method [3 – 6], F-expansion method [7], tanh-function method [8, 9], homotopy perturbation method [10], homotopy analysis method [11, 12], and so on.

In this paper, we consider the (2+1)-dimensional Chaffee-Infante equation in the following form:

$$u_{xt} + (-u_{xx} + \alpha u^3 - \alpha u)_x + \sigma u_{yy} = 0, \qquad (1)$$

where α and σ are arbitrary constants. The (2+1)-dimensional Chaffee-Infante equation is a well-known reaction duffing equation arising in mathematical physics (see [13] and references therein). Next, we consider the (1+1)-dimensional Chaffee-Infante equations [13]

$$u_t - u_{xx} = \alpha u(1 - u^2), \qquad (2)$$

where α is an arbitrary constant. The parameter α adjust the relative balance of the diffusion term and the nonlinear term. It is also called Newell-Whitehead equation when $\alpha = 1$.

The exp-function method [14] was proposed by He and Wu in 2006 to obtain exact solitary solutions and periodic solutions of nonlinear evolution equations, and has been successfully applied to many kinds of nonlinear partial differential equations [15 – 28]. All of these applications verified that the exp-function method is a straightforward and efficient method for finding exact solutions for nonlinear evolution equations. The main purpose of this paper is to obtain generalized new soliton solutions to (1) and (2) by using the exp-function method.

2. Solutions of (2+1)-Dimensional Chaffee-Infante Equation

In order to obtain the solution for (1), we consider the transformation $u = v(\eta)$, $\eta = kx + ly + \omega t$, where k, l, and ω are constants to be determined later. We can now rewrite the Chaffee-Infante equation (1) in the following nonlinear ordinary differential form:

$$k\omega v'' - k^3 v''' + 3\alpha k v^2 v' - \alpha k v' + \sigma l^2 v'' = 0, \qquad (3)$$

where the prime denotes the differential with respect to η. According to the exp-function method [5], we assume that the solution of (3) can be expressed in the form

$$v(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}, \tag{4}$$

where c, d, p, and q are positive integers which are unknown and have to be determined further, a_n and b_m are unknown constants.

(4) can be rewritten in an alternative form as

$$v(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)}. \tag{5}$$

In order to determine the values of c and p, we balance the linear term of highest order in (3) with the highest-order nonlinear term. By simple calculation, we have

$$v''' = \frac{c_1 \exp[(7p+c)\eta] + \cdots}{c_2 \exp(8p\eta) + \cdots} \tag{6}$$

and

$$\begin{aligned} v^2 v' &= \frac{c_3 \exp[(3p+3c)\eta] + \cdots}{c_4 \exp(6p\eta) + \cdots} \\ &= \frac{c_3 \exp[(5p+3c)\eta] + \cdots}{c_4 \exp(8p\eta) + \cdots}, \end{aligned} \tag{7}$$

where c_i are determined coefficients only for simplicity. Balancing highest-order of exp-function in (6) and (7), we obtain

$$7p + c = 5p + 3c \tag{8}$$

which gives

$$p = c. \tag{9}$$

Similarly to determine the values of d and q, we balance the linear term of lowest order in (3) with the lowest-order nonlinear term

$$v''' = \frac{\cdots + d_1 \exp[-(7q+d)\eta]}{\cdots + d_2 \exp[(-8q)\eta]} \tag{10}$$

and

$$\begin{aligned} v^2 v' &= \frac{\cdots + d_3 \exp[-(3q+3d)\eta]}{\cdots + d_4 \exp[(-8q)\eta]} \\ &= \frac{\cdots + d_3 \exp[-(5q+3d)\eta]}{\cdots + d_4 \exp[(-8q)\eta]}, \end{aligned} \tag{11}$$

where d_i are determined coefficients only for simplicity. Balancing lowest-order of exp-function in (10) and (11), we obtain

$$-(7q + d) = -(5q + 3d) \tag{12}$$

which gives

$$q = d. \tag{13}$$

Case 1: $p = c = 1, d = q = 1.$

We can freely choose the values of c and d, but the final solution does not strongly depend upon the choice of the values of c and d. For simplicity, we set $p = c = 1$, $b_1 = 1$, and $d = q = 1$, then (5) reduces to

$$v(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \tag{14}$$

Substituting (14) in (3) and using the Maple software, equating to zero the coefficients of all powers of $\exp(n\eta)$ gives a set of algebraic equations for a_1, a_0, a_{-1}, b_0, b_{-1}, k, l, and ω. Solving the systems of algebraic equations using Maple gives the following sets of non-trivial solutions:

$$\begin{aligned} \Big\{ & a_1 = 1, \quad a_0 = 0, \quad a_{-1} = 0, \quad b_0 = 0, \\ & b_{-1} = b_{-1}, \quad k = \frac{\delta_1}{2}, \quad l = l, \\ & \omega = -\frac{-3\alpha^2 + 16\delta_1 \sigma l^2}{4\alpha}, \quad \delta_1 = \pm\sqrt{\frac{\alpha}{2}} \Big\}, \end{aligned} \tag{15}$$

$$\begin{aligned} \Big\{ & a_1 = a_1, \quad a_0 = -\frac{b_0(2a_1^2 - 1)}{a_1}, \\ & a_{-1} = -\frac{(a_1^2 - 1)b_0^2}{8a_1}, \quad b_0 = b_0, \\ & b_{-1} = -\frac{(a_1^2 - 1)b_0^2}{8a_1^2}, \quad k = \pm\sqrt{3\alpha a_1^2 - \alpha}, \\ & l = l, \quad \omega = -\frac{\sigma l^2}{k} \Big\}, \end{aligned} \tag{16}$$

$$\begin{aligned} \Big\{ & a_1 = \pm\sqrt{\frac{1}{57}}, \quad a_0 = a_0, \quad a_{-1} = -\frac{133 a_1 a_0^2}{12}, \\ & b_0 = 0, \quad b_{-1} = -\frac{19a_0^2}{12}, \quad k = \pm\sqrt{\frac{6\alpha}{19}}, \\ & l = l, \quad \omega = -\frac{144\alpha^2 + 361 k\sigma l^2}{114} \Big\}, \end{aligned} \tag{17}$$

$$\begin{aligned} \Big\{ & a_1 = \pm\sqrt{\frac{1}{57}}, \quad a_0 = a_0, \quad a_{-1} = \frac{7N_1}{36N_2}, \\ & b_0 = b_0, \quad b_{-1} = \frac{19 N_1 a_1}{12 N_2}, \quad k = \pm\sqrt{\frac{6\alpha}{19}}, \\ & l = l, \quad \omega = -\frac{144\alpha^2 + 361 k\sigma l^2}{114\alpha} \Big\}, \end{aligned} \tag{18}$$

where

$$N_1 = 69 a_0 b_0^2 - 285 a_0^3 - 91 a_1 b_0^3 + 1539 a_1 a_0^2 b_0 \tag{19}$$

and

$$N_2 = 13b_0 + 285a_1a_0. \tag{20}$$

Substituting (15)–(18) in (14), we obtain the following soliton solutions of (1):

$$u_1(x,y,t) = \frac{\exp(\eta)}{\exp(\eta) + b_{-1}\exp(-\eta)}, \tag{21}$$

where $\eta = \frac{\delta_1}{2}x + ly - \left(\frac{-3\alpha^2+16\delta_1\sigma l^2}{4\alpha}\right)t$ and $\delta_1 = \pm\sqrt{\frac{\alpha}{2}}$,

$$u_2(x,y,t) = \frac{8a_1^3\exp(\eta) - 8b_0a_1(2a_1^2-1) - a_1(a_1^2-1)b_0^2\exp(-\eta)}{8a_1^2\exp(\eta) + 8a_1^2b_0 - (a_1^2-1)b_0^2\exp(-\eta)}, \tag{22}$$

where $\eta = \pm\sqrt{3\alpha a_1^2 - \alpha}x + ly - \left(\frac{\sigma l^2}{k}\right)t$,

$$u_3(x,y,t) = \frac{12a_1\exp(\eta) + 12a_0 - 133a_1a_0^2\exp(-\eta)}{12\exp(\eta) - 19a_0^2\exp(-\eta)}, \tag{23}$$

where $\eta = \pm\sqrt{\frac{6\alpha}{19}}x + ly - \left(\frac{144\alpha^2+361k\sigma l^2}{114}\right)t$ and $a_1 = \pm\sqrt{\frac{1}{57}}$,

$$u_4(x,y,t) = \frac{36a_1N_2\exp(\eta) + 36N_2a_0 + 7N_1\exp(-\eta)}{36N_2\exp(\eta) + 36N_2b_0 + 57N_1a_1\exp(-\eta)}, \tag{24}$$

where $\eta = \pm\sqrt{\frac{6\alpha}{19}}x + ly - \left(-\frac{144\alpha^2+361k\sigma l^2}{114\alpha}\right)t$ and N_1, N_2 are defined as in (19) and (20). Further, since b_{-1} is a free parameter, we take $b_{-1} = 1$, then (21) admits a new soliton solution of (1),

$$u(x,y,t) = \frac{1}{2}(1+\tanh\eta), \quad \text{where } \eta = \frac{\delta_1}{2}x + ly - \left(\frac{-3\alpha^2+16\delta_1\sigma l^2}{4\alpha}\right)t \quad \text{and } \delta_1 = \pm\sqrt{\frac{\alpha}{2}}. \tag{25}$$

Also when $a_1 = 1$, $b_0 = 1$, (22) admits a new soliton solution of (1),

$$u(x,y,t) = \tanh\frac{\eta}{2}, \quad \text{where } \eta = \pm\sqrt{2\alpha}x + ly - \left(\frac{\sigma l^2}{k}\right)t. \tag{26}$$

Case 2: $p = c = 2$, $d = q = 2$.

Now consider the case $p = c = 2$ and $d = q = 2$ with $a_2 = a_{-2} = b_1 = b_{-1} = 0$, $b_2 = 1$, under this case (5) can be expressed as

$$v(\eta) = \frac{a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{\exp(2\eta) + b_0 + b_{-2}\exp(-2\eta)}. \tag{27}$$

By the same calculation as illustrated in the previous case, we obtain

$$\left\{ a_1 = 1, \quad a_0 = 0, \quad a_{-1} = a_{-1}, \quad b_0 = \frac{8a_{-1}+a_1^3}{8a_1}, \quad b_{-2} = \frac{a_1a_{-1}}{8}, \quad k = \pm\sqrt{-\alpha}, \quad l = l, \quad \omega = \frac{\sigma l^2 k}{\alpha} \right\}. \tag{28}$$

Substituting (28) in (27), we obtain the following soliton solutions of (1):

$$u(x,t) = \frac{8a_1^2\exp(\eta) + 8a_1a_{-1}\exp(-\eta)}{8a_1\exp(2\eta) + 8a_{-1} + a_1^3 + a_1^2a_{-1}\exp(-2\eta)}, \tag{29}$$

where $\eta = kx + ly + \frac{\sigma l^2 k}{\alpha}t$, $k = \pm\sqrt{-\alpha}$.

3. Solutions of (1+1)-Dimensional Chaffee-Infante Equation

In this section, in order to obtain the solution of the Chaffee-Infante equation (2) we consider the transformation $u = v(\eta)$, $\eta = kx + \omega t$, which converts (2) into an ordinary differential equation of the form

$$\omega v' - k^2 v'' - \alpha v + \alpha v^3 = 0, \tag{30}$$

where the prime denotes the derivation with respect to η. By the same procedure as illustrated in Section 2, we can determine values of c and p by balancing v'' and v^2 in (30),

$$v'' = \frac{c_1\exp[(3p+c)\eta] + \cdots}{c_2\exp(4p\eta) + \cdots} \tag{31}$$

and

$$v^3 = \frac{c_3\exp[(3c)\eta] + \cdots}{c_4\exp(3p\eta) + \cdots} = \frac{c_3\exp[(p+3c)\eta] + \cdots}{c_4\exp(4p\eta) + \cdots}. \tag{32}$$

Balancing highest-order of exp-function in (31) and (32), we obtain

$$3p+c=p+3c \tag{33}$$

which gives

$$p=c. \tag{34}$$

By a similar derivation, balancing lowest-order of exp-function in (31) and (32), we obtain

$$-(3q+d)=-(q+3d) \tag{35}$$

which gives

$$q=d. \tag{36}$$

Case 1: $p=c=1, d=q=1$.

As mentioned in the previous section, the values of c and d can be freely chosen. For simplicity, we choose $p=c=1$, $b_1=1$, and $d=q=1$, then the trail function (5) becomes

$$v(\eta)=\frac{a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)}{\exp(\eta)+b_0+b_{-1}\exp(-\eta)}. \tag{37}$$

Substituting (37) in (30), equating to zero the coefficients of all powers of $\exp(n\eta)$ yields a set of algebraic equations for $a_1, a_0, a_{-1}, b_0, b_{-1}, k$, and ω. Solving the systems of algebraic equations using Maple, we obtain

$$\Big\{a_1=-1,\quad a_0=0,\quad a_{-1}=0,\quad b_0=0,\quad b_{-1}=b_{-1},\quad \delta_2=\pm\sqrt{\frac{\alpha}{2}},\quad k=\frac{\delta_2}{2},\quad \omega=\frac{3\alpha}{4},\Big\}, \tag{38}$$

$$\Big\{a_1=0,\quad a_0=0,\quad a_{-1}=-b_{-1},\quad b_0=0,\quad b_{-1}=b_{-1},\quad \delta_3=\pm\sqrt{\frac{\alpha}{2}},\quad k=\frac{\delta_3}{2},\quad \omega=-\frac{3\alpha}{4}\Big\}. \tag{39}$$

Substituting (38) and (39) in (37), we obtain the following soliton solutions of (2):

$$u_1(x,t)=\frac{\exp(-\eta)}{\exp(\eta)+b_{-1}\exp(-\eta)},\quad \text{where } \eta=\frac{\delta_2}{2}x+\frac{3\alpha}{4}t, \tag{40}$$

$$u_2(x,t)=\frac{-b_{-1}\exp(-\eta)}{\exp(\eta)+b_{-1}\exp(-\eta)},\quad \text{where } \eta=\frac{\delta_3}{2}x-\frac{3\alpha}{4}t. \tag{41}$$

Further, since b_{-1} is a free parameter, we choose $b_{-1}=1$, then (40) and (41) admit the following new soliton solutions:

$$u(x,t)=-\frac{1}{2}(1+\tanh(\eta)),\quad \text{where } \eta=\frac{\delta_2}{2}x+\frac{3\alpha}{4}t \text{ and } \delta_2=\pm\sqrt{\frac{\alpha}{2}}, \tag{42}$$

and

$$u(x,t)=-\frac{1}{2}(1-\tanh(\eta)),\quad \text{where } \eta=\frac{\delta_3}{2}x-\frac{3\alpha}{4}t \text{ and } \delta_3=\pm\sqrt{\frac{\alpha}{2}}, \tag{43}$$

respectively.

Case 2: $p=c=2, d=q=2$.

For simplicity, we set $p=c=2, b_2=1, b_1=b_{-1}=0$, and $d=q=1$, then by the same procedure as illustrated in Section 2, we obtain the following equation:

$$v(\eta)=\big[a_2\exp(2\eta)+a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)+a_{-2}\exp(-2\eta)\big]\cdot\big[\exp(2\eta)+b_0+b_{-2}\exp(-2\eta)\big]^{-1}. \tag{44}$$

By the same calculation as illustrated in the previous subsection, we obtain

$$\Big\{a_2=0,\quad a_1=1,\quad a_0=a_1^2,\quad a_{-1}=b_0a_1+a_1^3,\quad a_{-2}=b_0a_1^2+a_1^4, b_0=b_0,\quad b_{-2}=-a_1^4-b_0a_1^2,\quad k=\pm\sqrt{\frac{\alpha}{2}},\quad \omega=-\frac{3\alpha}{2}\Big\}, \tag{45}$$

$$\Big\{a_2=-1,\quad a_1=a_1,\quad a_0=-\frac{a_{-1}}{a_1},\quad a_{-1}=a_{-1},\quad a_{-2}=0,\quad b_0=-\frac{-a_{-1}+a_1^3}{a_1},\quad b_{-2}=-a_{-1}a_1,\quad k=\pm\sqrt{\frac{\alpha}{2}},\quad \omega=\frac{3\alpha}{2}\Big\}, \tag{46}$$

$$\Big\{ a_2 = 0, \quad a_1 = 0, \quad a_0 = a_0, \quad a_{-1} = 0, \\ b_0 = -\frac{-a_{-2} + a_0^2}{a_0}, \quad a_{-2} = a_{-2}, \quad b_{-2} = -a_{-2}, \\ \delta_4 = \pm\sqrt{\frac{\alpha}{2}}, \quad k = \frac{\delta_4}{2}, \quad \omega = -\frac{3\alpha}{4} \Big\}. \tag{47}$$

Substituting (45)–(47) in (44), we obtain the following soliton solutions of (2):

$$u(x,t) = \big[a_1 \exp(\eta) + a_1^2 + (b_0 a_1 + a_1^3)\exp(-\eta) \\ + (b_0 a_1^2 + a_1^4)\exp(-2\eta)\big] \\ \cdot \big[\exp(2\eta) + b_0 - (a_1^4 + b_0 a_1^2)\exp(-2\eta)\big]^{-1}, \tag{48}$$

where $\eta = \pm\sqrt{\frac{\alpha}{2}}x - \frac{3\alpha}{2}t$,

$$u(x,t) = \big[-a_1 \exp(2\eta) + a_1^2 \exp(\eta) - a_{-1} \\ + a_{-1} a_1 \exp(-\eta)\big]\big[a_1 \exp(2\eta) \\ - (-a_{-1} + a_1^3) - a_{-1} a_1^2 \exp(-2\eta)\big]^{-1}, \tag{49}$$

where $\eta = \pm\sqrt{\frac{\alpha}{2}}x + \frac{3\alpha}{2}t$,

$$u(x,t) = \big[a_0^2 + a_{-2} a_0 \exp(-2\eta)\big]\big[a_0 \exp(2\eta) \\ - (-a_{-2} + a_0^2) - a_{-2} a_0 \exp(-2\eta)\big]^{-1}, \tag{50}$$

where $\eta = \frac{\delta_4}{2}x - \frac{3\alpha}{4}t$ and $\delta_4 = \pm\sqrt{\frac{\alpha}{2}}$.

Case 3: $p = c = 3$, $d = q = 3$.

Now we consider the case $p = c = 3$, and $d = q = 3$ with $b_2 = 1$, $b_3 = b_1 = b_{-1} = 0$, under this case (5) can be expressed as

$$v(\eta) = \big[a_3 \exp(3\eta) + a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 \\ + a_{-1} \exp(-\eta) + a_{-2} \exp(-2\eta) + a_{-3} \exp(-3\eta)\big] \\ \cdot \big[\exp(2\eta) + b_0 + b_{-2} \exp(-2\eta) + b_{-3} \exp(-3\eta)\big]^{-1}. \tag{51}$$

By the same calculation as illustrated in the previous subsection, we obtain

$$\Big\{ a_3 = 0, \quad a_2 = 1, \quad a_1 = a_1, \quad a_0 = \frac{a_{-1}}{a_1}, \\ a_{-1} = a_{-1}, \quad a_{-2} = -\frac{b_{-3}}{a_1}, \quad a_{-3} = 0, \\ b_0 = \frac{a_{-1} - a_1^3}{a_1}, \quad b_{-2} = -\frac{b_{-3} + a_{-1} a_1^2}{a_1}, \\ b_{-3} = b_{-3}, \quad k = \pm\sqrt{\frac{\alpha}{2}}, \quad \omega = \frac{3\alpha}{2} \Big\}. \tag{52}$$

Substituting (52) in (51), we obtain the following soliton solutions of (2):

$$u(x,t) = \big[a_1 \exp(2\eta) + a_1^2 \exp(\eta) \\ + a_{-1} + a_1 a_{-1} \exp(-\eta) - b_{-3} \exp(-2\eta)\big] \\ \cdot \big[a_1 \exp(2\eta) + a_{-1} - a_1^3 \\ - (b_{-3} + a_{-1} a_1^2)\exp(-2\eta) + a_1 b_{-3} \exp(-3\eta)\big]^{-1}, \tag{53}$$

where $\eta = \pm\sqrt{\frac{\alpha}{2}}x + \frac{3\alpha}{2}t$.

4. Conclusion

In this paper, we have applied the exp-function method to obtain new generalized solitonary solutions of Chaffee-Infante equations. The correctness of these results is ensured by testing them on computer with the aid of the symbolic computation software Maple. More importantly, the exp-function method can give new and more general solutions when compared with most existing methods. This indicates the validity and great potential of the exp-function method in solving complicated solitary wave problems arising in mathematical physics.

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