Long Wavelength Flow Analysis in a Curved Channel

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This study is concerned with the peristaltic flow of a viscous fluid in a curved channel. Mathematically the problem is governed by two partial differential equations. Closed form solutions of the stream function, axial velocity, and pressure gradient are developed under long wavelength and low Reynolds number assumptions. The influence of curvature is analyzed on various flow quantities of interest.

Key words: Curved Channel; Peristalsis; Viscous Fluid; Modelling; Wave Frame.

1. Introduction

The peristaltic flows at low Reynolds number and long wavelength have gained increasing attention over the years. The interest in these flows is because of their wide range of applications either in engineering or medical sciences. Such applications include the urine transport from kidney to the bladder, bile from the gall bladder into the duodenum, spermatozoa in the ductus efferentes of the male reproductive tract, ovum in the fallopian tube, lymph in the lymphatic vessel, and so on.

One of the earliest studies on the peristaltic flows is due to Latham [1]. In 1969, Shapiro et al. [2] discussed the peristaltic flow in a planar channel by using long wavelength and low Reynolds number approximations. These studies have been subsequently followed up by several researchers in the field. Even now the analysis of peristaltic flows in a planar channel has been the subject of several recent papers [3 – 10]. These investigations have been made under one or more assumptions of long wavelength, low Reynolds number, small amplitude ratio, small wave number, etc.

It has been noticed from the existing literature that almost all the published papers on the peristalsis involve the flow geometries of planar channels / tubes. To the best of our knowledge, Sato et al. [11] have been the only research group who analyzed the two-dimensional peristaltic flow of a viscous fluid in a curved channel. However, no effort is made to investigate the peristaltic flow of a curved channel in the wave frame. Therefore, the aim of the present paper is to perform such a study. The considered geometry has potential applications in physiology since most of the glandular ducts are curved. The mathematical formulation in the wave frame is done for the peristaltic flow of a viscous fluid in a curved channel. The flow analysis is non-dimensionalized and then simplified under realistic assumptions of long wavelength and low Reynolds number. The results of various flow quantities are obtained and discussed. This study hopes to provide a fundamental basis for studying the peristaltic flow analysis in curved channels and in the wave frame.

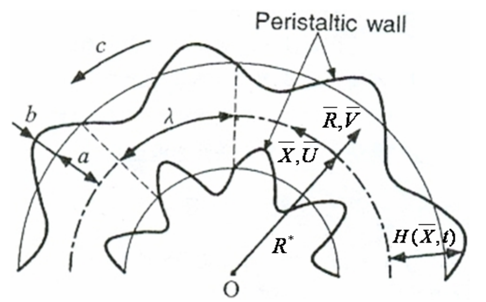

Fig. 1. Schematic diagram of the problem.

2. Problem Formulation

Let us consider a curved channel of half width a coiled in a circle with centre O and radius R^* (Fig. 1). The velocity components in radial ($\bar{R}$) and axial ($\bar{X}$) directions are $\bar{V}$ and $\bar{U}$, respectively. Sinusoidal waves of small amplitude b are imposed on the flexible walls of the channel. The peristaltic flow is induced because of the transverse deflections of the channel walls. The inertial effects are taken small. The geometries of channel walls are described as follows:

$$\bar{H}(\bar{X},\bar{t}) = a + b\sin\left[\frac{2\pi}{\lambda}(\bar{X}-c\bar{t})\right], \text{ upper wall,} \\ -\bar{H}(\bar{X},\bar{t}) = -a - b\sin\left[\frac{2\pi}{\lambda}(\bar{X}-c\bar{t})\right], \text{ lower wall.} \quad (1)$$

Here c is the speed and λ is the wavelength. λ is assumed to be a constant. However for more realistic situations it may depend upon time as the contraction and dilation in natural systems is in part on account on varying λ.

For the geometry under consideration, the dimensional equations of motion can be written as

$$\frac{\partial}{\partial\bar{R}}\{(\bar{R}+R^*)\bar{V}\} + R^*\frac{\partial\bar{U}}{\partial\bar{X}} = 0, \quad (2)$$

$$\frac{\partial\bar{V}}{\partial\bar{t}} + \bar{V}\frac{\partial\bar{V}}{\partial\bar{R}} + \frac{R^*\bar{U}}{R^*+\bar{R}}\frac{\partial\bar{V}}{\partial\bar{X}} - \frac{\bar{U}^2}{R^*+\bar{R}} = \\ -\frac{\partial\bar{p}}{\partial\bar{R}} + \nu\left[\frac{1}{R^*+\bar{R}}\frac{\partial}{\partial\bar{R}}\left\{(R^*+\bar{R})\frac{\partial\bar{V}}{\partial\bar{R}}\right\}\right. \quad (3) \\ \left. + \left(\frac{R^*}{R^*+\bar{R}}\right)^2\frac{\partial^2\bar{V}}{\partial\bar{X}^2} - \frac{\bar{V}}{(R^*+\bar{R})^2} - \frac{2R^*}{(R^*+\bar{R})^2}\frac{\partial\bar{U}}{\partial\bar{X}}\right],$$

$$\frac{\partial\bar{U}}{\partial\bar{t}} + \bar{V}\frac{\partial\bar{U}}{\partial\bar{R}} + \frac{R^*\bar{U}}{\bar{R}+R^*}\frac{\partial\bar{U}}{\partial\bar{X}} + \frac{\bar{U}\bar{V}}{\bar{R}+R^*} \\ = -\frac{R^*}{\bar{R}+R^*}\frac{\partial\bar{p}}{\partial\bar{X}} + \nu\left[\frac{1}{\bar{R}+R^*}\frac{\partial}{\partial\bar{R}}\left\{(\bar{R}+R^*)\frac{\partial\bar{U}}{\partial\bar{R}}\right\}\right. \quad (4) \\ \left. + \left(\frac{R^*}{\bar{R}+R^*}\right)^2\frac{\partial^2\bar{U}}{\partial\bar{X}^2} - \frac{\bar{U}}{(\bar{R}+R^*)^2} + \frac{2R^*}{(\bar{R}+R^*)^2}\frac{\partial\bar{V}}{\partial\bar{X}}\right].$$

In above equations $\bar{p}$ is the pressure, ν is the kinematic viscosity, and $\bar{t}$ is the time. In the laboratory frame $(\bar{R},\bar{X})$, the flow in the channel is unsteady. However, it can be treated as steady in a coordinate system $(\bar{r},\bar{x})$ moving with the wave speed c (wave frame). The two frames can be related in the following from:

$$\bar{x} = \bar{X} - c\bar{t}, \quad \bar{r} = \bar{R}, \quad \bar{u} = \bar{U} - c, \quad \bar{v} = \bar{V}, \quad (5)$$

where $\bar{v}$ and $\bar{u}$ are the velocity components along $\bar{r}$ and $\bar{x}$-directions in the wave frame. Now (2) – (4) can be casted in the wave frame as

$$\frac{\partial}{\partial\bar{r}}\{(\bar{r}+R^*)\bar{v}\} + R^*\frac{\partial\bar{u}}{\partial\bar{x}} = 0, \quad (6)$$

$$-c\frac{\partial\bar{v}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{v}}{\partial\bar{r}} + \frac{R^*(\bar{u}+c)}{R^*+\bar{r}}\frac{\partial\bar{v}}{\partial\bar{x}} - \frac{(\bar{u}+c)^2}{R^*+\bar{r}} = \\ -\frac{\partial\bar{p}}{\partial\bar{r}} + \nu\left[\frac{1}{R^*\bar{r}}\frac{\partial}{\partial\bar{r}}\left\{(R^*+\bar{r})\frac{\partial\bar{v}}{\partial\bar{r}}\right\}\right. \quad (7) \\ \left. + \left(\frac{R^*}{R^*+\bar{r}}\right)^2\frac{\partial^2\bar{v}}{\partial\bar{x}^2} - \frac{\bar{v}}{(R^*+\bar{r})^2} + \frac{2R^*}{(R^*+\bar{r})^2}\frac{\partial\bar{u}}{\partial\bar{x}}\right],$$

$$-c\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{u}}{\partial\bar{R}} + \frac{R^*(\bar{u}+c)}{\bar{R}+R^*}\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{(\bar{u}+c)\nu}{\bar{R}+R^*} = \\ -\frac{R^*}{\bar{R}+R^*}\frac{\partial\bar{p}}{\partial\bar{x}} + \nu\left[\frac{1}{\bar{r}+R^*}\frac{\partial}{\partial\bar{r}}\left\{(\bar{r}+R^*)\frac{\partial\bar{u}}{\partial\bar{r}}\right\}\right. \quad (8) \\ \left. + \left(\frac{R^*}{\bar{v}+R^*}\right)^2\frac{\partial^2\bar{u}}{(\bar{r}+R^*)^2} + \frac{2R^*}{(\bar{r}+R^*)^2}\frac{\partial\bar{v}}{\partial\bar{x}}\right].$$

If the dimensionless variables and stream function are defined as

$$x = \frac{2\pi\bar{x}}{\lambda}, \quad \eta = \frac{\bar{r}}{a}, \quad u = \frac{\bar{u}}{c}, \quad v = \frac{\bar{v}}{c}, \\ \text{Re} = \frac{\rho ca}{\mu}, P = \frac{2\pi a^2}{\lambda\mu c}\bar{p}, \quad h = \frac{\bar{H}}{a}, \quad (9) \\ \delta = \frac{2\pi a}{\lambda}, \quad k = \frac{R^*}{a},$$

$$u = \frac{\partial\psi}{\partial\eta}, \quad v = \delta\frac{k}{\eta+k}\frac{\partial\psi}{\partial x}, \quad (10)$$

then (6) is identically satisfied and (7) and (8), under long wavelength and low Reynolds number approximations [2 – 6, 8 – 11], can be reduced to the following dimensionless form:

$$\frac{\partial P}{\partial\eta} = 0, \quad (11)$$

$$-\frac{\partial P}{\partial x} - \frac{1}{k}\frac{\partial}{\partial\eta}\left\{(\eta+k)\frac{\partial^2\psi}{\partial\eta^2}\right\} \\ -\frac{1}{k(\eta+k)}\left(1-\frac{\partial\psi}{\partial\eta}\right) = 0. \quad (12)$$

From above equations, one obtains

$$\frac{\partial^2}{\partial\eta^2}\left\{(\eta+k)\frac{\partial^2\psi}{\partial\eta^2}\right\} \\ + \frac{\partial}{\partial\eta}\left\{\frac{1}{(\eta+k)}\left(1-\frac{\partial\psi}{\partial\eta}\right)\right\} = 0. \quad (13)$$

The dimensional volume flow rate in laboratory frame is defined as

$$Q = \int_{-H}^{H}\bar{U}\,\mathrm{d}\bar{R} \quad (14)$$

in which $\bar{H}$ is a function of $\bar{X}$ and $\bar{t}$. The above expression in wave frame becomes

$$F = \int_{-H}^{\bar{H}} \bar{u} \mathrm{d}\bar{r}, \tag{15}$$

where $\bar{H}$ is function of $\bar{x}$ alone. From (4) and (5) we can write

$$Q = F + 2c\bar{H}. \tag{16}$$

The time-averaged flow over a period T at a fixed position $\bar{X}$ is

$$\bar{Q} = \frac{1}{T}\int_0^T Q \mathrm{d}t. \tag{17}$$

Invoking (16) into (17) and then integrating one has

$$\bar{Q} = q + 2ac. \tag{18}$$

If we define the dimensionless mean flows Θ, in the laboratory frame, and q, in the wave frame, according to

$$\Theta = \frac{\bar{Q}}{ac}, \quad q = \frac{F}{ac}, \tag{19}$$

(18) reduces to

$$\Theta = q + 2, \tag{20}$$

in which

$$q = -\int_{-h}^{h} \frac{\partial \psi}{\partial \eta}\mathrm{d}\eta = -(\psi(h) - \psi(-h)). \tag{21}$$

Selecting $\psi(h) = -q/2$, we have $\psi(-h) = q/2$ and the appropriate boundary conditions in the wave frame are

$$\begin{aligned} &\psi = -\frac{q}{2}, \ \frac{\partial \psi}{\partial \eta} = 1 \ \text{ at } \ \eta = h = 1 + \phi \sin x, \\ &\psi = \frac{q}{2}, \ \frac{\partial \psi}{\partial \eta} = 1 \ \text{ at } \ \eta = -h = -1 - \phi \sin x, \end{aligned} \tag{22}$$

where $\phi = b/a$ is the amplitude ratio.

3. Solution of the Problem

The closed form solution of the problem consisting of (13) and boundary conditions (22) is

$$\begin{aligned} \psi = (\eta + k) + \frac{C_1}{2}\left\{(\ln(\eta+k) - 1)\frac{(\eta+k)^2}{2}\right\} \\ + C_2 \frac{(\eta+k)^2}{4} + C_3 \ln(\eta+k) + C_4. \end{aligned} \tag{23}$$

$$\begin{aligned} C_1 = &-8hk(2h+q)\Big[-4h^2k^2 + (h^2-k^2)^2\big(\ln(k-h)^2 \\ &+ \ln(k+h)^2\big) - 2(h^2-k^2)^2 \ln(k-h)\ln(k+h)\Big]^{-1}, \end{aligned}$$

$$\begin{aligned} C_2 = &-2(2h+q)\big(2hk + (k-h)^2 \ln(k-h) \\ &-(k+h)^2\ln(k+h)\big)\Big[-4h^2k^2 + (h^2-k^2)^2\big(\ln(k-h)^2 \\ &+ \ln(k+h)^2\big) - 2(h^2-k^2)^2\ln(k-h)\ln(k+h)\Big]^{-1}, \end{aligned}$$

$$\begin{aligned} C_3 = &(h^2-k^2)(2h+q)\ln\left(\frac{k-h}{k+h}\right)\Big[-4h^2k^2 \\ &+ (h^2-k^2)^2\big(\ln(k-h)^2 + \ln(k+h)^2\big) \\ &-2(h^2-k^2)^2\ln(k-h)\ln(k+h)\Big]^{-1}, \end{aligned}$$

$$\begin{aligned} C_4 = &2hk(2h^3 - 2hk^2 + h^2q + k^2q) + (h^2-k^2)^2 \\ &\cdot \ln\left(\frac{k-h}{k+h}\right)\big(-2h - q + (2(h+k)+q)\ln(k-h) \\ &+ (2h-2k+q)\ln(k+h)\big)\Big[2\big(-4h^2k^2 + (h^2-k^2)^2 \\ &\cdot \big(\ln(k-h)^2 + \ln(k+h)^2\big) \\ &-2\left(h^2-k^2\right)^2 \ln(k-h)\ln(k+h)\Big]^{-1}. \end{aligned}$$

From (10) the axial velocity turns out to be

$$\begin{aligned} u = 1 + \frac{C_1}{2}\Bigg[\frac{\eta+k}{2} + (\eta+k)\{\ln(\eta+k) - 1\} \\ + C_2\frac{\eta+k}{2} + \frac{C_3}{\eta+k}\Bigg], \end{aligned} \tag{24}$$

where C_{1-3} are defined above.

Upon making use of (12) the axial pressure gradient turned out to be

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}x} = &8h(2h+q)\Big[-4h^2k^2 + (h^2-k^2)^2\big(\ln(k-h)^2 \\ &+ \ln(k+h)^2\big) - 2(h^2-k^2)^2\ln(k-h)\ln(k+h)\Big]^{-1}. \end{aligned} \tag{25}$$

The dimensionless pressure rise over one wavelength is defined by

$$\Delta P_\lambda = \int_0^{2\pi} \frac{\mathrm{d}p}{\mathrm{d}x}\mathrm{d}x. \tag{26}$$

4. Results and Discussion

This section is divided into three subsections. Flow characteristics are described in first subsection. Second subsection is devoted to the discussion of pumping

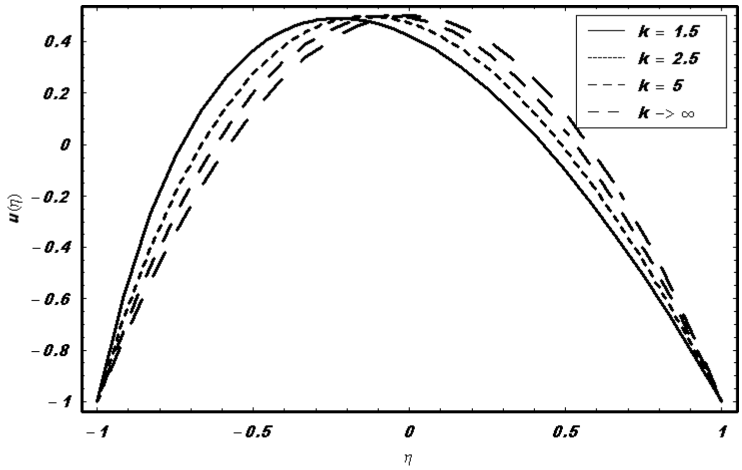

Fig. 2. Variation of $u(\eta)$ for different values of k with $\phi = 0.8$ and $\Theta = 2$.

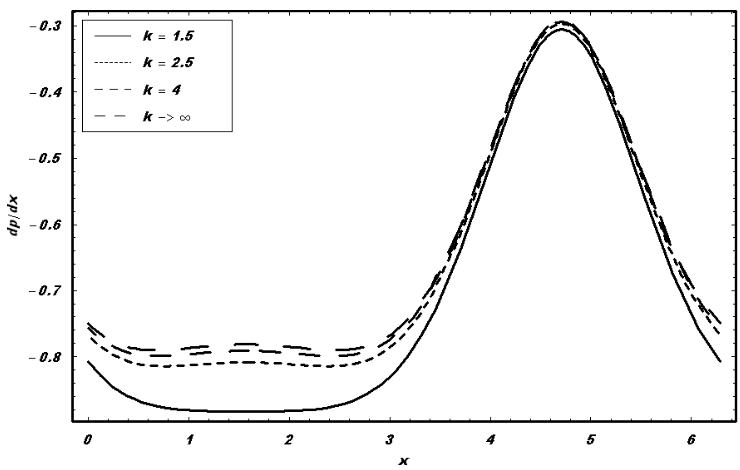

Fig. 3. Variation of $\mathrm{d}p/\mathrm{d}x$ for different values of k with $\phi = 0.2$ and $\Theta = 0.5$.

characteristics. In the last subsection the trapping phenomenon is illustrated. The analytical expressions of ψ, $u(\eta)$, and $\mathrm{d}p/\mathrm{d}x$ given in the previous section are used in this section to discuss these features of peristaltic motion. In the present analysis the extra parameter that comes into play in contrast with previous attempts on peristalsis is the curvature of the channel, i. e., k.

4.1. Flow Characteristics

The expression of u given by (24) can be used to discuss the flow characteristics. Therefore, its variation with η for different values of k is plotted in Figure 2.

It is noticed that for large values of k (i. e. for straight channel) the velocity profile is symmetric about the axis of the channel and the maxima occurs at $\eta = 0$. However, for small values of k (i. e. for curved channel) the profiles are not symmetric about $\eta = 0$ and maxima shifts towards the negative values of η. Furthermore it is observed from the computations that in the narrow part of the channel, the effects of curvature are not pronounced.

4.2. Pumping Characteristics

The pumping characteristics can be well described by studying the axial pressure gradient $\mathrm{d}p/\mathrm{d}x$ given by expression (25) and the pressure difference over one wavelength calculated from expression (26).

The variation of $\mathrm{d}p/\mathrm{d}x$ per wavelength for different values of k is seen in Figure 3. This figure depicts that the magnitude of $\mathrm{d}p/\mathrm{d}x$ decreases in going from curved to straight channel.

An interesting feature of peristalsis is pumping against the pressure rise. For such characteristics we

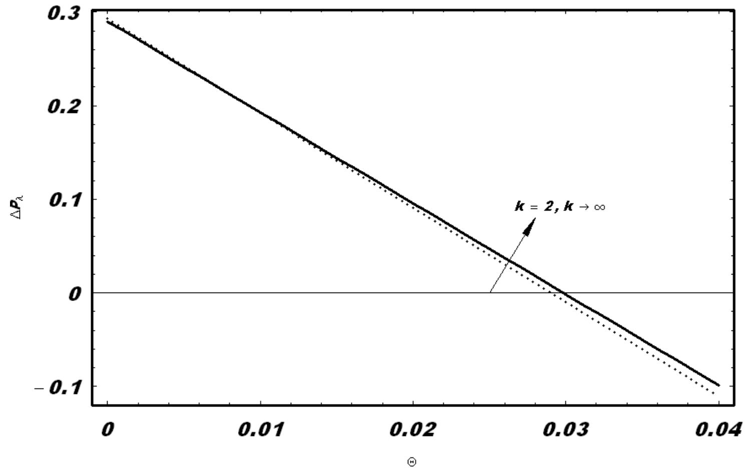

Fig. 4. Variation of ΔP_λ for different values of k with $\phi = 0.1$.

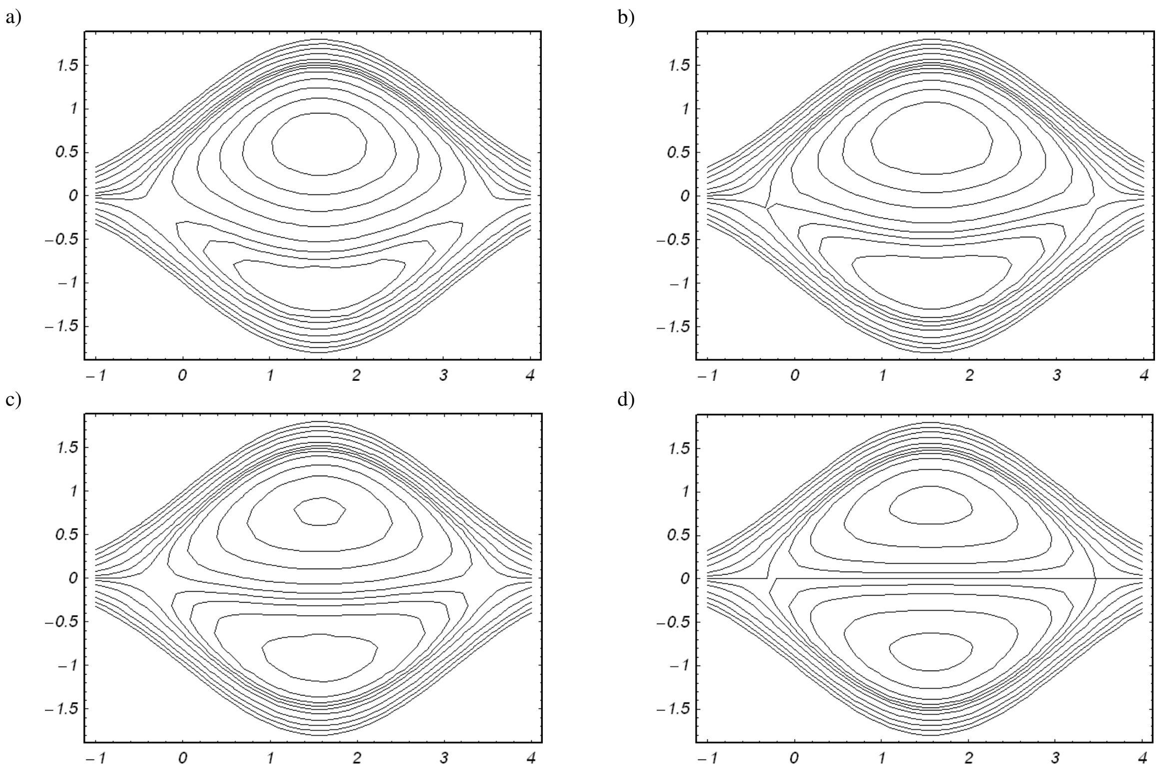

Fig. 5. Streamlines for $k = 3.5$ (panel a), $k = 5$ (panel b), $k = 10$ (panel c), and $k \to \infty$ (panel d). The other parameters are $\Theta = 1.5$ and $\phi = 0.8$.

have plotted pressure rise per wavelength ΔP_λ against dimensionless time mean flow rate Θ (Fig. 4) for different values of k. The maximum pressure rise against which peristalsis works as a pump, i. e., ΔP_λ for $\Theta = 0$, is denoted by P_0. When $\Delta P_\lambda > P_0$, then the flux is negative, i. e., against the peristaltic wave direction. The value of Θ corresponding to $\Delta P = 0$ (which is known as free pumping) is denoted by Θ_0. When $\Delta P_\lambda < 0$, the pressure assists the flow and this is known as co-pumping. The following information can be extracted from Figure 4:

- P_0 increases as one moves from straight to curved channel. This means that peristalsis has to work against greater pressure rise in curved channel as compared to flow in straight channel.

- The free pumping flux Θ_0 increases in going from curved to straight channel.
- In co-pumping similar to free pumping the pumping rate for straight channel is greater in magnitude as compared to curved channel.

4.3. Trapping

The analytical expression of ψ due to (23) is plotted in Figure 5 to discuss the trapping phenomenon for various values of k. In general the shape of streamlines is similar to that of the boundary wall in the wave frame. However, under certain conditions some of the streamline split and enclose a bolus, which moves as whole with the wave [2]. We observe from Figure 5 that for small values of k the bolus is not symmetric about $\eta = 0$ and is pushed towards the lower wall. However, as k increases the results of straight channel can be recovered.

5. Concluding Remarks

The problem of peristaltic motion of an incompressible viscous fluid is investigated in a curved rectangular channel. The problem is first modeled in wave frame under long wavelength and low Reynolds number approximation and then an exact solution is presented. The obtained solution is then used to discuss some interesting features of peristaltic motion. The results of straight channel can be recovered as a special case by choosing k large. The presented problem formulation is quite new in the regime of curved channel of which very little is said yet.

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