Exact Solutions for a Class of Nonlinear Singular Two-Point Boundary Value Problems: The Decomposition Method

Abd Elhalim Ebaid

Department of Mathematics, Faculty of Science, Tabuk University,
P. O. Box 741, Tabuki 71491, Saudi Arabia

Reprint requests to A. E. E.; E-mail: halimgamil@yahoo.com

Z. Naturforsch. **65a,** 145 – 150 (2010); received January 7, 2009 / revised May 23, 2009

Many problems in applied mathematics and engineering are usually formulated as singular two-point boundary value problems. A well-known fact is that the exact solutions in closed form of such problems were not obtained in many cases. In this paper, the exact solutions for a class of nonlinear singular two-point boundary value problems are obtained to the first time by using Adomian decomposition method.

Key words: Adomian Decomposition Method; Nonlinear Singular Two-Point Boundary Value Problems; Exact Solutions.

1. Introduction

Some of the most common problems in applied sciences and engineering are usually formulated as singular two-point boundary value problems of the form

$$(x^{\alpha}y')' = f(x,y), \quad 0 \le x \le 1. \tag{1}$$

Subject to the boundary conditions

$$y(0) = A \text{ and } y(1) = B, \tag{2}$$

where $\alpha \in (0,1]$, A and B are finite constants. For example, when $\alpha = 0$ and $f(x,y) = q(x)y^{-\sigma}$, (1) is known as the generalized Emden-Fowler equation with negative exponent and arises frequently in applied mathematics (see [1, 2] and the references cited therein). Also, when $\alpha = 0$ and $f(x,y) = -x^{-1/2}y^{\frac{3}{2}}$, (1) is known as Thomas-Fermi equation [3], given by the singular equation $y'' = x^{-1/2}y^{3/2}$, which arises in the study of the electrical potential in an atom. When $\alpha = p \in \{0,1,2\}$, another example is given by the singular equation $(x^p y')' = x^p f(y)$, which results from an analysis of heat conduction through a solid with heat generation. The function $f(y)$ represents the heat generation within the solid, y is the temperature, and the constant p is equal to 0, 1, or 2 depending on whether the solid is a plate, a cylinder or a sphere [4].

In recent years, the class of singular two-point boundary value problems (BVPs) modelled by (1) and (2) was studied by many mathematicians and a number of numerical methods [4 – 14] and analytical methods [15 – 18] have been proposed. Although these numerical methods have many advantages, a huge amount of computational work is required in obtaining accurate approximations. The most important notice here is that all the previous attempts during the last two decades were devoted only to obtain approximate solutions wether numerical or analytical. In this research, the exact solutions for the class (1) and (2) will be obtained to the first time by implementing the Adomian decomposition method (ADM) [19 – 21]. The ADM has been used extensively during the last two decades to solve effectively and easily a large class of linear and nonlinear ordinary and partial differential equations. However, a little attention was devoted for its application in solving the singular two-point boundary value problems. To the best of the author's knowledge, the only attempt to solve the singular two-point boundary value problems by using Adomian's method has been done recently by Inc and Evans [16]. They obtained an approximate solution for only one nonlinear example by using the ADM-Padé technique. Very recently, approximate solutions of linear singular two-point BVPs were obtained by Bataineh et al. [17] using the modified homotopy analysis method. Also in [18], Kanth and Aruna used another analytical method, the differential transformation method, to obtain the exact solutions for some linear singular two-point BVPs. Although these analytical methods [17, 18] were shown to be effective for solving a few linear examples, their applicability for nonlinear problems were not examined. So, we aim in this paper to show how to apply the decomposition method to obtain the exact solutions of these nonlinear singular two-point BVPs in a straightforward manner. Before doing this, let us begin by introducing the analysis of the method in the next section.

2. Analysis of Adomian Decomposition Method

In this section, the class of singular BVPs, (1) and (2), will be handled more easily, quickly, and elegantly by implementing the ADM rather than by the traditional methods for the exact solutions without making massive computational work. Let us begin our analysis by writing (1) in an operator form:

$$L_{xx}(y) = f(x,y), \tag{3}$$

where the linear differential operator L_{xx} is defined by

$$L_{xx}[.] = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\alpha}\frac{\mathrm{d}}{\mathrm{d}x}[.]\right). \tag{4}$$

The inverse operator L_{xx}^{-1} is therefore defined by

$$L_{xx}^{-1}[.] = \int_0^x \left(x^{-\alpha}\int_0^x [.]\mathrm{d}x\right)\mathrm{d}x. \tag{5}$$

Operating with L_{xx}^{-1} on (3), it then follows

$$y = y(0) + \int_0^x \left(x^{-\alpha}\int_0^x f(x,y)\mathrm{d}x\right)\mathrm{d}x. \tag{6}$$

Notice that only $y(0) = A$, is sufficient to carry out the solution and the other condition $y(1) = B$, can be used to show that the obtained solution satisfies this given condition. Assuming that $f(x,y) = r(x)g(y)$, (6) becomes

$$y = y(0) + \int_0^x \left(x^{-\alpha}\int_0^x r(x)g(y)\mathrm{d}x\right)\mathrm{d}x. \tag{7}$$

The ADM is based on decomposing y and the nonlinear term $g(y)$ as

$$y = \sum_{n=0}^{\infty} y_n, \quad g(y) = \sum_{n=0}^{\infty} A_n(y_0, y_1, \ldots, y_n), \tag{8}$$

where A_n are specially generated Adomian's polynomials for the specific nonlinearity and can be found from the formula

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} g\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}, \quad n \geq 0. \tag{9}$$

Inserting (8) into (7), it then follows

$$\sum_{n=0}^{\infty} y_n = y(0) + \sum_{n=0}^{\infty}\int_0^x \left(x^{-\alpha}\int_0^x r(x)A_n\mathrm{d}x\right)\mathrm{d}x. \tag{10}$$

According to the standard ADM, the solution can be computed by using the recurrence relation

$$\begin{aligned} y_0 &= y(0), \\ y_{n+1} &= \int_0^x \left(x^{-\alpha}\int_0^x [r(x)A_n]\mathrm{d}x\right)\mathrm{d}x, \quad n \geq 0. \end{aligned} \tag{11}$$

In the next section, we show that the algorithm (11) is very effective and powerful in obtaining the exact solutions for a wide class of nonlinear singular two-point boundary value problems in the form given by (1) and (2).

3. Applications and Exact Solutions

3.1. Example 3.1

Firstly, we consider the nonlinear singular equation [15, 16]:

$$y'' + \frac{1}{2x}y' = \mathrm{e}^y\left(\frac{1}{2} - \mathrm{e}^y\right), \tag{12}$$

subject to the boundary conditions

$$y(0) = \ln 2 \text{ and } y(1) = 0. \tag{13}$$

The numerical solutions of this nonlinear singular boundary value problem have been discussed by El-Sayed [15] and also by Inc and Evans [16]. In [15] the author established three iterative techniques to obtain the numerical solutions. While in [16], Inc and Evans used the standard ADM with Padé transformation to approximate the solution. These attempts, as mentioned above, were only to obtain approximate solutions. Here we show that we are able to obtain the exact solution by using the proposed algorithm (11). To do so, we firstly rewrite (12) in the form

$$\left(x^{1/2}y'\right)' = x^{1/2}\mathrm{e}^y\left(\frac{1}{2} - \mathrm{e}^y\right). \tag{14}$$

In order to apply algorithm (11) for this nonlinear singular second-order boundary value problem, we begin by using formula (9) to calculate the first few terms of Adomian's polynomials of the nonlinear term $e^y(\frac{1}{2} - e^y)$ as

$$\begin{aligned} A_0 &= e^{y_0}\left(\frac{1}{2} - e^{y_0}\right), \\ A_1 &= \frac{1}{2}e^{y_0}(1-4e^{y_0})y_1, \\ A_2 &= \frac{1}{4}e^{y_0}[y_1^2 + 2y_2 - 8e^{y_0}(y_1^2 + y_2)], \\ A_3 &= \frac{1}{12}e^{y_0}\left[y_1^3 + 6y_1y_2 + 6y_3 \right. \\ &\quad \left. - 8e^{y_0}(2y_1^3 + 6y_1y_2 + 3y_3)\right], \\ A_4 &= \frac{1}{48}e^{y_0}\left[y_1^4 + 12y_1^2y_2 + 12y_2^2 + 24y_1y_3 + 24y_4 \right. \\ &\quad \left. - 32e^{y_0}(y_1^4 + 6y_1^2y_2 + 3y_2^2 + 6y_1y_3 + 3y_4)\right]. \end{aligned} \tag{15}$$

Now, applying algorithm (11) on this singular BVP yields

$$\begin{aligned} y_0 &= \ln 2, \\ y_{n+1} &= \int_0^x \left(x^{-1/2}\int_0^x [x^{1/2}A_n]\mathrm{d}x\right)\mathrm{d}x, \quad n \geq 0. \end{aligned} \tag{16}$$

From this recurrence relation and the Adomian's polynomials given by (15), we can easily obtain

$$\begin{aligned} &y_0 = \ln 2, \quad y_1 = -x^2, \\ &y_2 = \frac{x^4}{2}, \quad y_3 = -\frac{x^6}{3}, \\ &y_4 = \frac{x^8}{4}, \quad y_5 = -\frac{x^{10}}{5}, \\ &\vdots \\ &y_n = \frac{(-1)^n x^{2n}}{n}, \quad n \geq 1. \end{aligned} \tag{17}$$

The solution is now given by

$$\begin{aligned} y &= \sum_{n=0}^{\infty} y_n = y_0 + \sum_{n=1}^{\infty} y_n = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n} \\ &= \ln 2 - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x^2)^n}{n} \\ &= \ln 2 - \ln(1+x^2) = \ln(\frac{2}{1+x^2}), \end{aligned} \tag{18}$$

which is the exact solution of (12) and (13). The numerical results for the absolute errors $|y(x)\text{-ADM-Padé}[5/5](x)|$ obtained in [16] and $|y(x) - \Phi_{10}(x)|$ are shown in Table 1. Note that the approximate solution $\Phi_{10} = \sum_{n=0}^{10} y_n$ is obtained through our algorithm (17). The numerical values in Table 1 show that a good approximation is achieved using small values of 10-terms of our algorithm without using Padé-approximant as made in [16].

Table 1.

x	$\lvert y(x)\text{-ADM-Padé}[5/5](x)\rvert$	$\lvert y(x) - \Phi_{10}(x)\rvert$
10^{-1}	4.432E-04	1.734E-17
10^{-2}	1.368E-04	0.000E-00
10^{-3}	1.300E-05	1.602E-17
10^{-4}	1.219E-06	1.077E-17
10^{-5}	2.659E-06	8.279E-18
10^{-6}	2.803E-06	2.212E-17

3.2. Example 3.2

We next consider the nonlinear singular equation [15]

$$y'' + \frac{1}{x}y' = y^3 - 3y^5, \tag{19}$$

subject to

$$y(0) = 1 \text{ and } y(1) = \frac{1}{\sqrt{2}}. \tag{20}$$

We also rewrite (19) in the form

$$(xy')' = x(y^3 - 3y^5). \tag{21}$$

This singular BVP has been also discussed by El-Sayed [15] by using the same iterative techniques to obtain the numerical solutions. Using algorithm (11) for this nonlinear singular boundary value problem, the solution can be elegantly computed by using the recursive scheme

$$\begin{aligned} y_0 &= 1, \\ y_{n+1} &= \int_0^x \left(x^{-1}\int_0^x [xA_n]\mathrm{d}x\right)\mathrm{d}x, \quad n \geq 0. \end{aligned} \tag{22}$$

The formula (9) gives the first few terms of Adomian's polynomials as

$$\begin{aligned} &A_0 = y_0^3 - 3y_0^5, \quad A_1 = 3(1-5y_0^2)y_0^2y_1, \\ &A_2 = 3[(1-10y_0^2)y_1^2 + y_0(1-5y_0^2)y_2]y_0, \\ &A_3 = (1-30y_0^2)y_1^3 + 6(1-10y_0^2)y_0y_1y_2 \\ &\quad + 3(1-5y_0^2)y_0^2y_3. \end{aligned} \tag{23}$$

Table 2.

x	Integral method [15]	$\lvert y(x) - \Phi_{10}(x)\rvert$
0.1	2.365E-05	1.110E-16
0.2	6.046E-06	1.110E-16
0.3	1.003E-06	4.859E-13
0.4	5.726E-07	2.565E-10
0.5	8.130E-08	3.235E-08

From the recursive relation (22) and Adomian's polynomials (23), we get

$$\begin{aligned} &y_0 = 1, \quad y_1 = \frac{-x^2}{2}, \\ &y_2 = \frac{3x^4}{8}, \quad y_3 = \frac{-5x^6}{16}, \\ &y_4 = \frac{35x^8}{128}, \quad y_5 = \frac{-63x^{10}}{256}, \\ &y_6 = \frac{231x^{12}}{1024}, \quad y_7 = \frac{-429x^{14}}{2048}, \\ &\vdots \end{aligned} \tag{24}$$

Hence, the solution in closed form is given by

$$\begin{aligned} y &= 1 - \frac{x^2}{2} + \frac{3x^4}{8} - \frac{5x^6}{16} + \frac{35x^8}{128} - \frac{63x^{10}}{256} + \cdots \\ &= \frac{1}{\sqrt{x^2+1}}. \end{aligned} \tag{25}$$

The numerical results for the absolute errors obtained via integral method [15] and $|y(x) - \Phi_{10}(x)|$ are shown in Table 2. Here, the approximate solution $\Phi_{10} = \sum_{n=0}^{10} y_n$ is obtained via algorithm (24). The numerical results in Table 2 show that very small errors are achieved using only 10-terms of our algorithm.

3.3. Example 3.3

Finally, we consider the nonlinear singular boundary value problem [11]

$$y'' + \frac{1}{x}y' + \nu e^y = 0. \tag{26}$$

This equation has the exact solution

$$\begin{aligned} &y = 2\ln\left(\frac{B+1}{Bx^2+1}\right), \\ &\text{where } B = \frac{(8-2\nu) \pm \sqrt{(8-2\nu)^2 - 4\nu^2}}{2\nu}. \end{aligned} \tag{27}$$

For simplicity, we consider (26) when $\nu = 2$. In this case, the exact solution of (26) is given by

$$y = 2\ln\left(\frac{2}{x^2+1}\right). \tag{28}$$

So in this example we consider the nonlinear singular BVP

$$y'' + \frac{1}{x}y' + 2e^y = 0, \tag{29}$$

subject to

$$y(0) = 2\ln 2 \text{ and } y(1) = 0. \tag{30}$$

Kumar [11], applied a three-point finite difference method based on a uniform mesh to obtain approximate numerical solution for this nonlinear singular boundary value problem. However, a huge amount of computational work bas been done to obtain such solution. We aim here to confirm that algorithm (11), not only used in a straightforward manner, but also leads directly to the exact solution. Before doing so, we rewrite (29) in the form

$$(xy')' = -xe^y. \tag{31}$$

Following the same analysis of the previous examples, the solution of this singular BVP can be elegantly computed by using the recursive scheme

$$\begin{aligned} &y_0 = 2\ln 2, \\ &y_{n+1} = -2\int_0^x \left(x^{-1}\int_0^x [xA_n]\mathrm{d}x\right)\mathrm{d}x, \quad n \geq 0, \end{aligned} \tag{32}$$

with Adomian's polynomials of the nonlinear term e^y given as

$$\begin{aligned} &A_0 = e^{y_0}, \quad A_1 = e^{y_0}y_1, \quad A_2 = \frac{1}{2}e^{y_0}(y_1^2 + 2y_2), \\ &A_3 = \frac{1}{6}e^{y_0}(y_1^3 + 6y_1y_2 + 6y_3), \\ &A_4 = \frac{1}{24}e^{y_0}(y_1^4 + 12y_1^2y_2 + 24y_1y_3 + 12y_2^2 + 24y_4). \end{aligned} \tag{33}$$

By using the recursive relation (32) and the first few terms of Adomian's polynomials (33), we can easily obtain

$$\begin{aligned} &y_0 = 2\ln 2, \quad y_1 = -2x^2, \\ &y_2 = x^4 = 2\left(\frac{x^4}{2}\right), \quad y_3 = 2\left(\frac{-x^6}{3}\right), \end{aligned}$$

Table 3.

x	Three-point F-D method [15]	$\lvert y(x)-\Phi_{10}(x)\rvert$
0.1	6.199E-09	0.000E-00
0.2	5.720E-10	0.000E-00
0.3	8.616E-10	3.552E-15
0.4	3.288E-09	6.032E-12
0.5	2.0966E-09	1.860E-09

$$y_4=\frac{x^8}{2}=2\left(\frac{x^8}{4}\right),\quad y_5=2\left(\frac{-x^{10}}{5}\right),$$
$$y_6=\frac{x^{12}}{3}=2\left(\frac{x^{12}}{6}\right),\quad y_7=2\left(\frac{-x^{14}}{7}\right),$$
$$\vdots$$
$$y_n=2\left[\frac{(-1)^n(x^2)^n}{n}\right],\quad n\geq 1. \tag{34}$$

Now, the solution can be put in the closed form:

$$\begin{aligned} y&=\sum_{n=0}^{\infty}y_n=y_0+\sum_{n=1}^{\infty}y_n\\ &=2\ln 2+2\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{n}\\ &=2\ln 2-2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x^2)^n}{n}\\ &=2\ln 2-2\ln(1+x^2)=2\ln\left(\frac{2}{1+x^2}\right), \end{aligned} \tag{35}$$

which is also the exact solution of (29) and (30).

In Table 3, the numerical results for the absolute errors obtained by the method in [11] and $|y(x)-\Phi_{10}(x)|$ are shown, where Φ_{10} is obtained by algorithm (34). Although the numerical results of the absolute errors obtained by the method in [11] are very small, our absolute errors are still better without a huge amount of computational work as in [11].

4. Remarks

Initially, we note that the class given by (1)–(2) involve only regular-singular points. So, algorithm (11) is valid in this case only. Hence, for problems with irregular-singularity, algorithm (11) should be changed. To make this point as clear as possible, let us consider a general class of singular two-point BVPs,

$$y''+\frac{\lambda}{x^\mu}y'=f(x,y),\ \mu>1,\ (\lambda \text{ is a finite constant}), \tag{36}$$

subject to the boundary conditions given by (2). It should be noted that with $\mu>1$, the singular point $x=0$ becomes an irregular-singular point. In order to establish the new algorithm, we firstly transform (36) into a new equivalent form given by

$$(x^\mu y')'=[\mu x^{\mu-1}-\lambda]y'+x^\mu f(x,y). \tag{37}$$

Then, we define the inverse operator L_{xx}^{-1} as

$$L_{xx}^{-1}[.]=\int_0^x\left(x^{-\mu}\int_0^x[.]\mathrm{d}x\right)\mathrm{d}x. \tag{38}$$

Operating with L_{xx}^{-1} on (37) and using the same analysis of Section 2, we obtain the general algorithm

$$\begin{aligned} &y_0=y(0),\\ &y_{n+1}=\int_0^x\left[x^{-\mu}\int_0^x(\mu x^{\mu-1}-\lambda)y_n'\mathrm{d}x\right]\mathrm{d}x\\ &\qquad+\int_0^x\left(x^{-\mu}\int_0^x[x^\mu r(x)A_n]\mathrm{d}x\right)\mathrm{d}x,\\ &n\geq 0. \end{aligned} \tag{39}$$

Finally, we note that algorithm (39) is valid for the class (36) under the following conditions:

(i) The integral $\int_0^x\left[x^{-\mu}\int_0^x(\mu x^{\mu-1}-\lambda)y_n'\mathrm{d}x\right]\mathrm{d}x$, exists $\forall\, n\geq 1$.

(ii) The integral $\int_0^x(x^{-\mu}\int_0^x[x^\mu r(x)A_n]\mathrm{d}x)\,\mathrm{d}x$, exists $\forall\, n\geq 1$.

5. Conclusions

Based on Adomian's method, an efficient approach is proposed in this work to solve effectively and easily a class of nonlinear singular two-point BVPs. Moreover, the proposed approach not only used in a straightforward manner, but also requires less computational works in comparison with the other methods.

Acknowledgements

The author would like to thank the referees for their comments and discussions.

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