Reconstruction of the Diffusion Operator from Nodal Data

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In this paper, we deal with the inverse problem of reconstructing the diffusion equation on a finite interval. We prove that a dense subset of nodal points uniquely determine the boundary conditions and the coefficients of the diffusion equation. We also provide constructive procedure for them.

Key words: Diffusion Operator; Inverse Nodal Problem; Reconstruction Formula.
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1. Introduction

Inverse spectral problems consist in recovering operators from their spectral characteristics. Such problems play an important role in mathematics and have many applications in natural sciences (see, for example, [1 – 6]). In 1988, the inverse nodal problem was posed and solved for Sturm-Liouville problems by J. R. McLaughlin [7], who showed that the knowledge of a dense subset of nodal points of the eigenfunctions alone can determine the potential function of the Sturm-Liouville problem up to a constant. This is the so-called inverse nodal problem [8]. Inverse nodal problems consist in constructing operators from the given nodes (zeros) of the eigenfunctions. Recently, some authors have reconstructed the potential function for generalizations of the Sturm-Liouville problem from the nodal points (for example, [7 – 20]).

In this paper we concern ourselves with the reconstruction of the diffusion equation from nodal data. The novelty of this paper lies in the use of a dense subset of nodal points for the eigenfunctions as the given spectra data for the reconstruction of the diffusion equation.

2. Main Results

The problem of describing the interactions between colliding particles is of fundamental interest in physics. One is interested in collisions of two spinless particles, and it is supposed that the s-wave scattering matrix and the s-wave binding energies are exactly known from collision experiments. With a radial static potential $V(x)$ the s-wave Schrödinger equation is written as

$$y'' + [E - V(E,x)]y = 0,$$

where $V(E,x)$ has the following form of the energy dependence:

$$V(E,x) = 2\sqrt{E}P(x) + Q(x).$$

Before giving the main results of this work, we mention some properties of the diffusion equation. The diffusion operator is written as

$$L[y] \triangleq -y'' + [q(x) + 2\lambda p(x)]y, \quad x \in [0,\pi],$$

where the function $q(x) \in L^2[0,\pi]$ and $p(x) \in W_2^1[0,\pi]$. Let λ_n be the spectrum of the problem

$$\begin{cases} L[y] = \lambda^2 y, \\ y'(0,\lambda) - hy(0,\lambda) = 0, \\ y'(\pi,\lambda) + Hy(\pi,\lambda) = 0, \\ h|y(0)|^2 + H|y(\pi)|^2 \\ + \int_0^\pi [|y'(x)|^2 + q(x)|y(x)|^2]\mathrm{d}x > 0, \quad h, H \in \mathbf{R}^+ \end{cases} \tag{1}$$

and $y(x,\lambda_n)$ the eigenfunction corresponding to the eigenvalue λ_n.

It is well known that the sequence $\{\lambda_n : n \in \mathbf{Z}\}$ satisfies the classical asymptotic form [21, 22]

$$\lambda_n = n + c_0 + \frac{c_1}{n} + \frac{c_{1,n}}{n}, \tag{2}$$

where $\sum_n |c_{1,n}|^2 < \infty$ and

$$c_0 = \frac{1}{\pi}\int_0^\pi p(x)\mathrm{d}x, \quad c_1 = \frac{1}{\pi}\left\{h + H + \frac{1}{2}\int_0^\pi [q(x)+p^2(x)]\mathrm{d}x\right\}. \tag{3}$$

The solution of the equation $L[y] = \lambda^2 y$ with the initial conditions $y(0) = 1$ and $y'(0) = h$ is

$$y(x,\lambda) = \cos\left[\sqrt{\lambda}x - \alpha(x)\right] + \int_0^x A(x,t)\cos\left(\sqrt{\lambda}t\right)\mathrm{d}t + \int_0^x B(x,t)\sin\left(\sqrt{\lambda}t\right)\mathrm{d}t,$$

where the kernels $A(x,t), B(x,t) \in L^1([0,\pi]\times[0,\pi])$ and

$$\alpha(x) = \int_0^x p(t)\mathrm{d}t. \tag{4}$$

Let $0 < x_1^n < \ldots < x_j^n < \ldots < x_{n-1}^n < \pi$ be the nodal points of the n-th eigenfunction $y(x,\lambda_n)$. In other words, $y(x_j^n,\lambda_n) = 0$, $j = 1,2,\cdots,n-1$. Let be $I_j^n = (x_j^n, x_{j+1}^n)$ and the nodal length l_j^n be

$$l_j^n = x_{j+1}^n - x_j^n. \tag{5}$$

Define $x_0^n = 0$ and $x_n^n = \pi$. We also define the function $j_n(x)$ to be the largest index j such that $0 \le x_j^n \le x$. Thus, $j = j_n(x)$ if and only if $x \in [x_j^n, x_{j+1}^n)$.

Define $X \triangleq \{x_j^n\}_{n\ge 0, j=\overline{1,n-1}}$. X is called the set of nodal points of the diffusion operator (1).

Under the condition that h, H, and $p(x)$ in (1) are known the paper [13] gave the reconstruction of the potential function $q(x)$ of the diffusion operator by nodal data.

When we solve the inverse problem from the spectra data, the obvious question occurs: What if h, H, and $p(x)$, $q(x)$ in (1) are all unknown? Our motivation in considering nodal points of eigenfunctions as data is our desire to obtain "more" information on the diffusion operator. In this paper, we prove that a dense subset X of nodal data uniquely determine the coefficients $q(x)$ and $p(x)$, and h and H in (1).

Define

$$F_j^n \triangleq n x_j^n - \left(j - \frac{1}{2}\right)\pi, \tag{6}$$

$$G_j^n \triangleq n^2\left[x_j^n - \frac{(j-\frac{1}{2})\pi}{n} - \frac{1}{n}\int_0^{x_j^n} p(t)\mathrm{d}t + \frac{(j-\frac{1}{2})\pi c_0}{n^2} - \frac{1}{n}\int_0^{x_j^n} p(t)\cos[(2n+2c_0)t]\mathrm{d}t\right], \tag{7}$$

$$H_j^n \triangleq \frac{n^2}{\pi}\left[l_j^n - \frac{\pi}{n}\right], \tag{8}$$

$$K_j^n \triangleq \frac{2(h+H)}{\pi} - 2c_0^2 + \frac{1}{\pi}\int_0^x p^2(x)\mathrm{d}x + \frac{2n^3}{\pi}\left[l_j^n - \frac{\pi}{n} - \frac{1}{n}\int_{x_j^n}^{x_{j+1}^n} p(t)\mathrm{d}t + \frac{\pi c_0}{n^2} - \frac{1}{n}\int_{x_j^n}^{x_{j+1}^n} p(t)\cos[(2n+2c_0)t]\mathrm{d}t + \frac{c_0}{n^2}\int_{x_j^n}^{x_{j+1}^n} p(t)\mathrm{d}t\right]. \tag{9}$$

The main theorems are the following.

Theorem 2.1. For $x \in [0,\pi]$. Let $\{x_j^n\} \subset X$ be chosen such that $\lim_{n\to\infty} x_j^n = x$. Then the following finite limits exist

$$g_1(x) \triangleq \lim_{n\to\infty} F_j^n, \quad g_2(x) \triangleq \lim_{n\to\infty} G_j^n \tag{10}$$

and

$$g_1(x) = \int_0^x p(x)\mathrm{d}x - c_0 x, \quad g_2(x) = -\frac{h}{2} + \frac{1}{2}\int_0^x q(x)\mathrm{d}x - c_0\int_0^x p(x)\mathrm{d}x + (c_0^2 - c_1)x. \tag{11}$$

Let us now formulate a uniqueness theorem and provide a constructive procedure for the solution of the inverse nodal problem.

Theorem 2.2. Let $X_0 \subset X$ be a subset of nodal points which is dense on $(0,\pi)$. Then, the specification of X_0 uniquely determines $p(x) - \frac{1}{\pi}\int_0^\pi p(x)\mathrm{d}x$ and $q(x) - \frac{1}{\pi}\int_0^\pi q(x)\mathrm{d}x$ on $(0,\pi)$, and the coefficients h and H of the boundary conditions in (1). $p(x) - \frac{1}{\pi}\int_0^\pi p(x)\mathrm{d}x$ and $q(x) - \frac{1}{\pi}\int_0^\pi q(x)\mathrm{d}x$, and the numbers h and H can be constructed via the formulae

$$p(x) - \frac{1}{\pi}\int_0^\pi p(x)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}x}g_1(x), \quad h = -2g_2(0), \quad H = -\frac{3h}{2} - \frac{1}{2}\int_0^\pi p^2(x)\mathrm{d}x - g_2(\pi), \tag{12}$$

and

$$q(x)-\frac{1}{\pi}\int_0^\pi q(x)\mathrm{d}x=2\frac{\mathrm{d}}{\mathrm{d}x}g_2(x)+\frac{2h+2H}{\pi}+2c_0p(x)+\frac{1}{\pi}\int_0^\pi p^2(x)\mathrm{d}x-2c_0^2, \quad (13)$$

where $g_1(x)$ and $g_2(x)$ are calculated by (11).

Theorem 2.3. Given $x\in[0,\pi]$. Let $\{x_j^n\}\subset X$ be chosen such that $\lim_{n\to\infty}x_j^n=x$. Then H_j^n converges to $p(x)-\frac{1}{\pi}\int_0^\pi p(x)\mathrm{d}x$ a. e. $x\in[0,\pi]$ and in $L^1(0,\pi)$-norm, and K_j^n converges to $q(x)-\frac{1}{\pi}\int_0^\pi q(x)\mathrm{d}x$ a. e. $x\in[0,\pi]$ and in $L^1(0,\pi)$-norm.

Using only the nodal data and the constants, namely $\frac{1}{\pi}\int_0^\pi p(x)\mathrm{d}x$ and $\frac{1}{\pi}\int_0^\pi q(x)\mathrm{d}x$, we can reconstruct these unknown coefficients. Our reconstruction formulae are direct and automatically imply the uniqueness of this inverse problem.

3. Proofs

Before proving the theorems we shall derive some results that will be used later on to establish our principal results.

For nodal points x_j^n (the zero points of the n-th eigenfunction), the asymptotic formula for nodal points ($n\to\infty$) follows from [13]

$$x_j^n=\frac{(j-\frac{1}{2})\pi}{\lambda_n}-\frac{h}{2\lambda_n^2}+\frac{1}{2\lambda_n^2}\int_0^{x_j^n}[1+\cos(2\lambda_n t)]\,[q(t)+2\lambda_n p(t)]\,\mathrm{d}t+O\left(\frac{1}{\lambda_n^4}\right). \quad (14)$$

Taking (2) into account and using Taylor's expansions for $(1+x)^\alpha$ and $\cos x$, we shall obtain the refinement of nodal points. Simple calculations show that

$$\lambda_n^{-1}=\left(n+c_0+\frac{c_1}{n}+\frac{c_{1,n}}{n}\right)^{-1}=\frac{1}{n}-\frac{c_0}{n^2}+\frac{c_0^2-c_1}{n^3}-\frac{c_{1,n}}{n^3}+O\left(\frac{1}{n^4}\right), \quad (15)$$

$$\lambda_n^{-2}=\left(n+c_0+\frac{c_1}{n}+\frac{c_{1,n}}{n}\right)^{-2}=\frac{1}{n^2}-\frac{2c_0}{n^3}+O\left(\frac{1}{n^4}\right), \quad (16)$$

and

$$\begin{aligned}\cos(2\lambda_n t)&=\cos\left[2\left(n+c_0+\frac{c_1}{n}+\frac{c_{1,n}}{n}\right)t\right]\\&=\cos[(2n+2c_0)t]\cos\left[\left(\frac{2c_1}{n}+\frac{2c_{1,n}}{n}\right)t\right]-\sin[(2n+2c_0)t]\sin\left[\left(\frac{2c_1}{n}+\frac{2c_{1,n}}{n}\right)t\right]\\&=\cos[(2n+2c_0)t]\left[1-\frac{2c_1^2t^2}{n^2}+o\left(\frac{1}{n^2}\right)\right]-\sin[(2n+2c_0)t]\left[\left(\frac{2c_1}{n}+\frac{2c_{1,n}}{n}\right)t+o\left(\frac{1}{n^2}\right)\right]\\&=\cos[(2n+2c_0)t]-\frac{2c_1^2t^2\cos[(2n+2c_0)t]}{n^2}-\frac{2(c_1+c_{1,n})t\sin[(2n+2c_0)t]}{n}+o\left(\frac{1}{n^2}\right).\end{aligned} \quad (17)$$

Plugging these expressions for (15), (16), and (17) into (14) and using the Riemann-Lebesgue Lemma and Lemma 3.1, we obtain asymptotic formulae for nodal points as $n\to\infty$ uniformly in j, $j=\overline{1,n-1}$:

$$\begin{aligned}x_j^n&=\frac{(j-\frac{1}{2})\pi}{n}+\frac{1}{n}\int_0^{x_j^n}p(t)\mathrm{d}t-\frac{(j-\frac{1}{2})\pi c_0}{n^2}\\&+\frac{1}{n}\int_0^{x_j^n}p(t)\cos[(2n+2c_0)t]\mathrm{d}t-\frac{h}{2n^2}-\frac{c_0}{n^2}\int_0^{x_j^n}p(t)\mathrm{d}t\\&+\frac{1}{2n^2}\int_0^{x_j^n}q(t)\mathrm{d}t-\frac{c_0}{n^2}\int_0^{x_j^n}p(t)\cos[(2n+2c_0)t]\mathrm{d}t\\&+\frac{1}{2n^2}\int_0^{x_j^n}q(t)\cos[(2n+2c_0)t]\mathrm{d}t\\&-\frac{2(c_1+c_{1,n})}{n^2}\int_0^{x_j^n}tp(t)\sin[(2n+2c_0)t]\mathrm{d}t\\&+\frac{(c_0^2-c_1-c_{1,n})(j-\frac{1}{2})\pi}{n^3}+O\left(\frac{1}{n^3}\right).\end{aligned} \quad (18)$$

We note that the set X is dense on $(0,\pi)$.

By the asymptotic formula (18) for nodal points above, using the Riemann-Lebesgue Lemma and Lemma 3.1, we can obtain the asymptotic expansion of nodal lengths as follows:

$$l_j^n=\frac{\pi}{n}+\frac{1}{n}\int_{x_j^n}^{x_{j+1}^n}p(t)\mathrm{d}t-\frac{\pi c_0}{n^2}+\frac{1}{n}\int_{x_j^n}^{x_{j+1}^n}p(t)\cos[(2n+2c_0)t]\mathrm{d}t+O\left(\frac{1}{n^3}\right). \quad (19)$$

In the above results, the order estimate is independent of j. As a result,

$$l_j^n = \frac{\pi}{n} + o\left(\frac{1}{n}\right). \quad (20)$$

Lemma 3.1. Suppose that $f \in L^1(0,\pi)$. Then for $x \in (0,\pi)$, with $j = j_n(x)$,

$$\lim_{n\to\infty} \int_{x_j^n}^{x_{j+1}^n} f(t)\mathrm{d}t = 0, \quad (21)$$

$$\lim_{n\to\infty} \frac{n}{\pi} \int_{x_j^n}^{x_{j+1}^n} f(t)\cos[(2n+2c_0)t]\mathrm{d}t = 0, \quad (22)$$

and

$$\lim_{n\to\infty} \frac{n}{\pi} \int_{x_j^n}^{x_{j+1}^n} f(t)\sin[(2n+2c_0)t]\mathrm{d}t = 0. \quad (23)$$

Proof. We first show that the result (21) holds if $f \in C[0,\pi]$. Let $M = \max_{x\in[0,\pi]} |f(x)|$. Then

$$\left|\int_{x_j^n}^{x_{j+1}^n} f(t)\mathrm{d}t\right| \le M(x_{j+1}^n - x_j^n) = Ml_j^n = O\left(\frac{1}{n}\right).$$

Therefore, if $f \in C[0,\pi]$, then $\left|\int_{x_j^n}^{x_{j+1}^n} f(t)\mathrm{d}t\right|$ can be arbitrarily small for large n which implies (21) is true.

Since $C[0,\pi]$ is dense in $L^1(0,\pi)$, for any $f \in L^1(0,\pi)$ there exists a sequence $f_k \in C(0,\pi)$ that converges to f in $L^1(0,\pi)$. Now

$$\left|\int_{x_j^n}^{x_{j+1}^n} f(t)\mathrm{d}t\right| \le \left|\int_{x_j^n}^{x_{j+1}^n} (f(t)-f_k(t))\mathrm{d}t\right| + \left|\int_{x_j^n}^{x_{j+1}^n} f_k(t)\mathrm{d}t\right|. \quad (24)$$

For any $\varepsilon > 0$, fix k large enough and n large enough such that the first term of the right-hand side in (24) is small than ε, together with (20). For all n large enough, the last term is small than ε by above. Hence, for $f \in L^1(0,\pi)$, (21) is true. Using the same method above, we can verify that (22) and (23) are true. The proof is finished. □

Lemma 3.2. Suppose that $f \in L^1(0,\pi)$. Then, with $j = j_n(x)$,

$$\lim_{n\to\infty} \frac{n}{\pi} \int_{x_j^n}^{x_{j+1}^n} f(t)\mathrm{d}t = f(x) \text{ a.e. } x \in [0,\pi] \quad (25)$$

and

$$\left\|\frac{n}{\pi} \int_{x_j^n}^{x_{j+1}^n} f(t)\mathrm{d}t - f(x)\right\|_1 \to 0 \text{ as } n \to \infty. \quad (26)$$

Proof. Note that

$$\begin{aligned}
&\frac{n}{\pi} \int_{x_j^n}^{x_{j+1}^n} f(t)\mathrm{d}t \\
&= \frac{n-\lambda_n}{\pi} \int_{x_j^n}^{x_{j+1}^n} f(t)\mathrm{d}t + \frac{\lambda_n}{\pi} \int_{x_j^n}^{x_{j+1}^n} f(t)\mathrm{d}t \\
&= O(1) \int_{x_j^n}^{x_{j+1}^n} f(t)\mathrm{d}t + \frac{\lambda_n}{\pi} \int_{x_j^n}^{x_{j+1}^n} f(t)\mathrm{d}t.
\end{aligned}$$

Applying the result in [10, 12]: Suppose that $f \in L^1(0,\pi)$, with $j = j_n(x)$, there hold

$$\lim_{n\to\infty} \frac{\lambda_n}{\pi} \int_{x_j^n}^{x_{j+1}^n} f(t)\mathrm{d}t = f(x) \text{ a.e. } x \in [0,\pi]$$

and

$$\left\|\frac{\lambda_n}{\pi} \int_{x_j^n}^{x_{j+1}^n} f(t)\mathrm{d}t - f(x)\right\|_1 \to 0 \quad \text{as } n \to \infty,$$

we conclude that (25) and (26) hold. The proof is complete. □

Now we can give the proofs of the theorems in this paper.

Proof of Theorem 2.1 Using the asymptotic expansion for nodal points in (18) we get

$$\begin{aligned}
F_j^n = &\int_0^{x_j^n} p(t)\mathrm{d}t - \frac{(j-\frac{1}{2})\pi c_0}{n} \\
&+ \int_0^{x_j^n} p(t)\cos[(2n+2c_0)t]\mathrm{d}t \\
&- \frac{h}{2n} - \frac{c_0}{n}\int_0^{x_j^n} p(t)\mathrm{d}t + \frac{1}{2n}\int_0^{x_j^n} q(t)\mathrm{d}t \\
&- \frac{c_0}{n}\int_0^{x_j^n} p(t)\cos[(2n+2c_0)t]\mathrm{d}t \\
&+ \frac{1}{2n}\int_0^{x_j^n} q(t)\cos[(2n+2c_0)t]\mathrm{d}t \\
&- \frac{2(c_1+c_{1,n})}{n}\int_0^{x_j^n} t\,p(t)\sin[(2n+2c_0)t]\mathrm{d}t \\
&+ \frac{(c_0^2-c_1-c_{1,n})(j-\frac{1}{2})\pi}{n^2} + O\left(\frac{1}{n^2}\right)
\end{aligned}$$

and

$$G_j^n = -\frac{h}{2} - c_0 \int_0^{x_j^n} p(t)\mathrm{d}t + \frac{1}{2}\int_0^{x_j^n} q(t)\mathrm{d}t + \frac{(c_0^2 - c_1 - c_{1,n})(j-\frac{1}{2})\pi}{n} - c_0 \int_0^{x_j^n} p(t)\cos[(2n+2c_0)t]\mathrm{d}t + \frac{1}{2}\int_0^{x_j^n} q(t)\cos[(2n+2c_0)t]\mathrm{d}t - 2(c_1 + c_{1,n})\int_0^{x_j^n} t p(t)\sin[(2n+2c_0)t]\mathrm{d}t + O\left(\frac{1}{n}\right).$$

Using the Riemann-Lebesgue Lemma we obtain as $n \to \infty$

$$F_j^n = \int_0^{x_j^n} p(t)\mathrm{d}t - \frac{(j-\frac{1}{2})\pi c_0}{n} + o(1) \tag{27}$$

and

$$G_j^n = -\frac{h}{2} - c_0 \int_0^{x_j^n} p(t)\mathrm{d}t + \frac{1}{2}\int_0^{x_j^n} q(t)\mathrm{d}t + \frac{(c_0^2 - c_1)(j-\frac{1}{2})\pi}{n} + o(1). \tag{28}$$

Also, the fact that $\lim_{n\to\infty} x_j^n = x$ implies that $\lim_{n\to\infty} \frac{(j-\frac{1}{2})\pi}{n} = x$ from (18), and

$$\lim_{n\to\infty} \int_0^{x_j^n} f(x)\mathrm{d}x = \int_0^x f(x)\mathrm{d}x.$$

From (27) and (28) it follows that

$$\lim_{n\to\infty} F_j^n \triangleq g_1(x) = \int_0^x p(x)\mathrm{d}x - c_0 x \tag{29}$$

and

$$\lim_{n\to\infty} G_j^n \triangleq g_2(x) = -\frac{h}{2} + \frac{1}{2}\int_0^x q(x)\mathrm{d}x - c_0\int_0^x p(x)\mathrm{d}x + (c_0^2 - c_1)x. \tag{30}$$

This proves the theorem. □

Proof of Theorem 2.2

Now given a nodal subset X_0, by Theorem 2.1 we can build up the reconstruction formulae.

Formulae (12) and (13) can be derived directly from (29) and (30) stepwise. We present the following procedure.

Step 1: Taking derivatives in (29) we obtain

$$p(x) - \frac{1}{\pi}\int_0^\pi p(x)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}x} g_1(x). \tag{31}$$

Step 2: Taking $x = 0$ in (30), it follows $g_2(0) = -\frac{h}{2}$. Hence,

$$h = -2g_2(0). \tag{32}$$

Step 3: After h and p are reconstructed, taking $x = \pi$ in (30), it follows

$$\begin{aligned} g_2(\pi) &= -\frac{h}{2} + \frac{1}{2}\int_0^\pi q(x)\mathrm{d}x - c_0\int_0^\pi p(x)\mathrm{d}x + (c_0^2 - c_1)\pi \\ &= -\frac{h}{2} + \frac{1}{2}\int_0^\pi q(x)\mathrm{d}x - \pi c_0^2 + \pi c_0^2 - h - H - \frac{1}{2}\int_0^\pi [q(x) + p^2(x)]\mathrm{d}x \\ &= -\frac{3h}{2} - H - \frac{1}{2}\int_0^\pi p^2(x)\mathrm{d}x, \end{aligned}$$

which yields

$$H = -\frac{3h}{2} - \frac{1}{2}\int_0^\pi p^2(x)\mathrm{d}x - g_2(\pi). \tag{33}$$

Step 4: After h, H, and p are reconstructed, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} g_2(x) &= \frac{1}{2}q(x) - c_0 p(x) + c_0^2 - c_1 \\ &= \frac{1}{2}q(x) - c_0 p(x) + c_0^2 - \frac{h+H}{\pi} - \frac{1}{2\pi}\int_0^\pi q(x)\mathrm{d}x - \frac{1}{2\pi}\int_0^\pi p^2(x)\mathrm{d}x, \end{aligned}$$

thus,

$$q(x) - \frac{1}{\pi}\int_0^\pi q(x)\mathrm{d}x = 2\frac{\mathrm{d}}{\mathrm{d}x} g_2(x) + \frac{2h+2H}{\pi} + 2c_0 p(x) + \frac{1}{\pi}\int_0^\pi p^2(x)\mathrm{d}x - 2c_0^2. \tag{34}$$

Since each nodal data only determine a set of reconstruction formulae which only depend on nodal data, the uniqueness holds obviously. □

Proof of Theorem 2.3

Using the asymptotic expansion for the nodal length in (19) and using the Riemann-Lebesgue Lemma and Lemma 3.1 we obtain as $n \to \infty$

$$H_j^n = \frac{n}{\pi}\int_{x_j^n}^{x_{j+1}^n} p(t)\mathrm{d}t - c_0 + o(1) \tag{35}$$

and

$$K_j^n = \frac{n}{\pi}\int_{x_j^n}^{x_{j+1}^n} q(t)\mathrm{d}t - \frac{1}{\pi}\int_0^{\pi} q(x)\mathrm{d}x + o(1). \tag{36}$$

From this we get

$$\left|H_j^n - \left[p(x) - \frac{1}{\pi}\int_0^{\pi} p(x)\mathrm{d}x\right]\right| \le \left|\frac{n}{\pi}\int_{x_j^n}^{x_{j+1}^n} p(t)\mathrm{d}t - p(x)\right| + o(1).$$

Using Lemmas 3.1 and 3.2 yields

$$\lim_{n\to\infty}\left|H_j^n - \left[p(x) - \frac{1}{\pi}\int_0^{\pi} p(x)\mathrm{d}x\right]\right| = 0 \text{ a. e. } x \in [0,\pi].$$

Moreover,

$$\int_0^{\pi}\left|H_j^n - \left[p(x) - \frac{1}{\pi}\int_0^{\pi} p(x)\mathrm{d}x\right]\right|\mathrm{d}x \le \int_0^{\pi}\left|\frac{n}{\pi}\int_{x_j^n}^{x_{j+1}^n} p(t)\mathrm{d}t - p(x)\right|\mathrm{d}x + o(1).$$

Lemma 3.2 tells us

$$\lim_{n\to\infty}\int_0^{\pi}\left|H_j^n - \left[p(x) - \frac{1}{\pi}\int_0^{\pi} p(x)\mathrm{d}x\right]\right|\mathrm{d}x = 0.$$

Thus, H_j^n converges to $p(x) - \frac{1}{\pi}\int_0^{\pi} p(x)\mathrm{d}x$ a. e. $x \in [0,\pi]$ and in $L^1(0,\pi)$ norm.

Also, we get

$$\left|K_j^n - \left[q(x) - \frac{1}{\pi}\int_0^{\pi} q(x)\mathrm{d}x\right]\right| \le \left|\frac{n}{\pi}\int_{x_j^n}^{x_{j+1}^n} q(t)\mathrm{d}t - q(x)\right| + o(1).$$

Using Lemma 3.2 yields

$$\lim_{n\to\infty}\left|K_j^n(x) - \left[q(x) - \frac{1}{\pi}\int_0^{\pi} q(x)\mathrm{d}x\right]\right| = 0$$

a. e. $x \in [0,\pi]$.

Moreover,

$$\int_0^{\pi}\left|K_j^n - \left[q(x) - \frac{1}{\pi}\int_0^{\pi} q(x)\mathrm{d}x\right]\right|\mathrm{d}x \le \int_0^{\pi}\left|\frac{n}{\pi}\int_{x_j^n}^{x_{j+1}^n} q(t)\mathrm{d}t - q(x)\right|\mathrm{d}x + o(1).$$

Lemma 3.2 tells us

$$\lim_{n\to\infty}\int_0^{\pi}\left|K_j^n - \left[q(x) - \frac{1}{\pi}\int_0^{\pi} q(x)\mathrm{d}x\right]\right|\mathrm{d}x = 0.$$

Therefore K_j^n converges to $q(x) - \frac{1}{\pi}\int_0^{\pi} q(x)\mathrm{d}x$ a. e. $x \in [0,\pi]$ and in $L^1(0,\pi)$ norm. The proof of theorem is complete. □

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