Homotopy Perturbation Method for a Reliable Analytic Treatment of some Evolution Equations

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Z. Naturforsch. **65a,** 59 – 64 (2010); received April 27, 2009 / revised June 12, 2009

In this paper, we suggest a new reliable application of He's homotopy perturbation method to study some evolution equations. The new application accelerates the rapid convergence of the series solutions and is used for analytic treatment of these equations. Some illustrative examples are given to further highlight the reliability and flexibility of the homotopy perturbation method.

Key words: Homotopy Perturbation Method; Evolution Equations; Homotopy; Exact Solution.
PACS numbers: 02.30.Jr; 04.20.Jb

1. Introduction

A new perturbation method called the homotopy perturbation method (HPM) [1 – 4] was proposed by Ji-Huan He in 1999, which is a coupling of the traditional perturbation method and homotopy in topology. The traditional perturbation methods are based on assuming a small parameter, and the approximate solutions obtained by those methods, in most cases, are valid only for small values of the small parameter even though many of nonlinear problems having strong nonlinearity have no small parameters at all. The homotopy perturbation method, without demanding a small parameter in the equations, deforms continuously to a simple problem which is easily solved. In this method, the solution is given in an infinite series, which usually converges rapidly to an accurate solution. The approximations obtained by the HPM are universally valid for small parameters, but also for very large parameters. This new method was further developed and improved by He, and a considerable amount of research has been conducted in applying this method to various kinds of linear and nonlinear problems, such as nonlinear oscillators with discontinuities and conservative nonlinear oscillators [5 – 8], nonlinear wave equations [9], limit cycle and bifurcations [10 – 13], nonlinear boundary value problems [14 – 15], asymptotology [16], integro-differential equation [17 – 19], non-Newtonian flow [20 – 21], systems of differential equations and stiff systems [22 – 24], differential-difference equations [25 – 26], nonlinear Korteweg-de Vries and Burgers equations [27 – 30], Schrödinger equations [31], the Cauchy reaction-diffusion problem [32], and many other subjects [33 – 45]. All of these applications verified that this method is a very effective and powerful tool for solving both linear and nonlinear problems. For a complete survey on the HPM and its applications, see [34 – 36] and the references therein.

It should be noted that even though it is well perceived that the HPM solution usually produces excellent results, there may be some special cases in which the simple homotopy equation may yield a divergent series solution. In order to apply the HPM successfully, it is important to construct a suitable homotopy equation, for which the simple problem involved should outline the basic character of the solution. A suitable choice of a homotopy equation will yield excellent results with just a few iterations. Some explanation of this issue is well addressed by Ji-Huan He in his series of tutorial articles in [37 – 39]. In this paper we aim to extend this issue and make a further investigation in this regard. A reliable application of the homotopy perturbation method will be proposed. This will provide rapid convergence of the series solutions as compared with the standard HPM as well as a reliable analytic treatment of some evolution equations. Several examples are tested, and the obtained results suggest that the newly developed technique could lead to an effective complement for the application of the HPM, and to further reveal the power of the HPM over existing numerical methods.

0932–0784 / 10 / 0100–0059 $ 06.00 © 2010 Verlag der Zeitschrift für Naturforschung, Tübingen · http://znaturforsch.com

2. Basic Idea of He's Homotopy Perturbation Method

To illustrate the homotopy perturbation method for solving differential equations, He [1–2] considered the following nonlinear differential equation:

$$A(u) = f(r), \quad r \in \Omega, \tag{1}$$

subject to the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω, and $\frac{\partial}{\partial n}$ denotes differentiation along the normal vector drawn outwards from Ω. The operator A can generally be divided into two parts, M and N. Therefore, (1) can be rewritten as

$$M(u) + N(u) = f(r), \quad r \in \Omega. \tag{3}$$

He [1–2] constructed a homotopy $v(r, p) : \Omega \times [0,1] \to \Re$ which satisfies

$$H(v, p) = (1-p)[M(v) - M(u_0)] + p[A(v) - f(r)] = 0, \tag{4}$$

which is equivalent to

$$H(v, p) = M(v) - M(u_0) + pM(u_0) + p[N(v) - f(r)] = 0, \tag{5}$$

where $p \in [0,1]$ is an embedding parameter, and u_0 is an initial approximation of (3). Obviously, we have

$$H(v,0) = M(v) - M(u_0) = 0, \quad H(v,1) = A(v) - f(r) = 0. \tag{6}$$

The changing process of p from zero to unity is just that of $H(v, p)$ from $M(v) - M(u_0)$ to $A(v) - f(r)$. In topology, this is called deformation and $M(v) - M(u_0)$ and $A(v) - f(r)$ are called homotopic. According to the homotopy perturbation method, the parameter p is used as a small parameter, and the solution of (4) can be expressed as a series in p in the form

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots. \tag{7}$$

When $p \to 1$, (4) corresponds to the original one, (1), and (7) becomes the approximate solution of (1), i. e.,

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots. \tag{8}$$

If (1) admits a unique solution, then this method produces the unique solution. If (1) does not possess a unique solution, the HPM will give a solution among many other possible solutions. The convergence of the series in (8) is discussed by He in [1–2].

3. An Application of He's Homotopy Perturbation Method

The homotopy perturbation method successfully and easily solves a variety of partial differential equations. However, the application of the HPM with some homotopy to some evolution equations may lead to divergent series solutions, this is certainly not a desirable situation. In the following examples we illustrate these cases.

Example 3.1 Consider the problem [40]

$$u_t + u_x = 2u_{xxt}, \quad -\infty < x < +\infty, \quad t > 0, \tag{9}$$

$$u(x,0) = \mathrm{e}^{-x}. \tag{10}$$

We construct the following homotopy:

$$u_t = p(2u_{xxt} - u_x). \tag{11}$$

Substituting (7) into (11) and equating coefficients of like powers of p, we obtain

$$p^0 : \frac{\partial v_0}{\partial t} = 0, \quad v_0(x,0) = \mathrm{e}^{-x}, \tag{12}$$

$$p^1 : \frac{\partial v_1}{\partial t} = 2\frac{\partial^3 v_0}{\partial x^2 \partial t} - \frac{\partial v_0}{\partial x}, \quad v_1(x,0) = 0, \tag{13}$$

$$\vdots$$

$$p^n : \frac{\partial v_n}{\partial t} = 2\frac{\partial^3 v_{n-1}}{\partial x^2 \partial t} - \frac{\partial v_{n-1}}{\partial x}, \quad v_n(x,0) = 0. \tag{14}$$

The solutions of the above equations are given by

$$\begin{aligned} v_0(x,t) &= \mathrm{e}^{-x}, \\ v_1(x,t) &= t\mathrm{e}^{-x}, \\ v_2(x,t) &= \left(2t + \frac{1}{2}t^2\right)\mathrm{e}^{-x}, \\ v_3(x,t) &= \left(4t + 2t^2 + \frac{1}{6}t^3\right)\mathrm{e}^{-x}, \\ &\vdots \end{aligned} \tag{15}$$

According to the HPM, we can conclude that

$$u = \sum_{n=0}^{\infty} v_n. \tag{16}$$

This result is in full agreement with that obtained by using the standard Adomian decomposition method (ADM) and the sequence on the right-hand side of (16) can be shown to be divergent for any value of t (see [40] for more detail). It should be remarked that the homotopy constructed by

$$u_t + u_x = 2pu_{xxt} \tag{17}$$

also gives a divergent series.

Example 3.2 Consider the problem [40]

$$u_t + 2\frac{\partial^4 u}{\partial x^4} = \frac{\partial^3 u}{\partial x^2 \partial t}, \quad -\infty < x < +\infty, \quad t > 0, \tag{18}$$

$$u(x,0) = \sin x. \tag{19}$$

We construct the following homotopy:

$$u_t = p\left[\frac{\partial^3 u}{\partial x^2 \partial t} - 2\frac{\partial^4 u}{\partial x^4}\right]. \tag{20}$$

Substituting (7) into (20) and equating coefficients of like powers of p, we obtain

$$\begin{aligned}
&p^0 : \frac{\partial v_0}{\partial t} = 0, \quad v_0(x,0) = \sin x,\\
&p^1 : \frac{\partial v_1}{\partial t} = \frac{\partial^3 v_0}{\partial x^2 \partial t} - 2\frac{\partial^4 v_0}{\partial x^4}, \quad v_1(x,0) = 0,\\
&\vdots\\
&p^n : \frac{\partial v_n}{\partial t} = \frac{\partial^3 v_{n-1}}{\partial x^2 \partial t} - 2\frac{\partial^4 v_{n-1}}{\partial x^4}, \quad v_n(x,0) = 0.
\end{aligned} \tag{21}$$

The solutions of the above equations are given by

$$\begin{aligned}
&v_0(x,t) = \sin x,\\
&v_1(x,t) = -2t\sin x,\\
&v_2(x,t) = (2t^2 + 2t)\sin x,\\
&v_3(x,t) = -\left(2t + 4t^2 + \frac{4}{3}t^3\right)\sin x,\\
&\vdots
\end{aligned} \tag{22}$$

The solution given by

$$u = \sum_{n=0}^{\infty} v_n \tag{23}$$

is in full agreement with that obtained by the standard Adomian decomposition method, and the sequence on the right-hand side of (23) can be shown to be divergent for any value of t (see [40] for more detail). It should be remarked that another possible homotopy defined by

$$u_t + 2\frac{\partial^4 u}{\partial x^4} = p\frac{\partial^3 u}{\partial x^2 \partial t}, \tag{24}$$

results in a complicated divergent series. In [40] a modified ADM is considered to deal with this situation, which gives the solution

$$u(x,t) = e^{-t-x}. \tag{25}$$

However, their approach is involved more in applying to cases than in general situation.

4. A Reliable Application of He's Homotopy Perturbation Method

In the discussed problems in the previous section, we observed that the HPM in some special cases may yield a divergent series solution. However, these situations can be easily overcome by considering appropriate homotopy with a converging parameter as will be seen in the following, this is revealing the reliability of the HPM.

Example 4.1 Consider the same problem as in Example 3.1:

$$u_t + u_x = 2u_{xxt}, \quad -\infty < x < +\infty, \quad t > 0, \tag{26}$$

$$u(x,0) = e^{-x}. \tag{27}$$

To solve this problem, we construct the following homotopy:

$$u_t + \beta u_x = p\left[2u_{xxt} + (\beta - 1)u_x\right], \tag{28}$$

where β is a real number further to be determined.

Substituting (7) into (28) and equating coefficients of like powers of p, we obtain

$$p^0 : \frac{\partial v_0}{\partial t} + \beta\frac{\partial v_0}{\partial x} = 0, \quad v_0(x,0) = e^{-x}, \tag{29}$$

$$p^1 : \frac{\partial v_1}{\partial t} + \beta\frac{\partial v_1}{\partial x} = 2\frac{\partial^3 v_0}{\partial x^2 \partial t} + (\beta - 1)\frac{\partial v_0}{\partial x}, \; v_1(x,0) = 0, \tag{30}$$

$$\vdots$$

$$p^n: \frac{\partial v_n}{\partial t} + \beta \frac{\partial v_n}{\partial x} = 2\frac{\partial^3 v_{n-1}}{\partial x^2 \partial t} + (\beta - 1)\frac{\partial v_{n-1}}{\partial x}, \quad v_n(x,0) = 0. \tag{31}$$

The solution of (29) is

$$v_0(x,t) = \mathrm{e}^{\beta t - x}, \tag{32}$$

(30) then becomes

$$\frac{\partial v_1}{\partial t} + \beta \frac{\partial v_1}{\partial x} = (\beta + 1)\mathrm{e}^{\beta t - x}, \quad v_1(x,0) = 0. \tag{33}$$

Now we take $\beta = -1$ so that the solution of (33) is $v_1(x,t) = 0$. In this case, $v_n(x,t) = 0$, $n \geq 2$. This in turn gives the exact solution in a closed form

$$u(x,t) = \sum_{n=0}^{\infty} v_n = \mathrm{e}^{-t-x}. \tag{34}$$

Example 4.2 Consider the same problem as in Example 3.2:

$$u_t + 2\frac{\partial^4 u}{\partial x^4} = \frac{\partial^3 u}{\partial x^2 \partial t}, \quad -\infty < x < +\infty, \quad t > 0, \tag{35}$$

$$u(x,0) = \sin x. \tag{36}$$

To solve this problem, we construct the following homotopy:

$$u_t + \beta \frac{\partial^4 u}{\partial x^4} = p\left[\frac{\partial^3 u}{\partial x^2 \partial t} + (\beta - 2)\frac{\partial^4 u}{\partial x^4}\right], \tag{37}$$

where β is a real number further to be determined.

Substituting (7) into (37) and equating coefficients of like powers of p, we obtain

$$p^0: \frac{\partial v_0}{\partial t} + \beta \frac{\partial^4 v_0}{\partial x^4} = 0, \quad v_0(x,0) = \sin x, \tag{38}$$

$$p^1: \frac{\partial v_1}{\partial t} + \beta \frac{\partial^4 v_1}{\partial x^4} = \frac{\partial^3 v_0}{\partial x^2 \partial t} + (\beta - 2)\frac{\partial^4 v_0}{\partial x^4}, \quad v_1(x,0) = 0, \tag{39}$$

$$\vdots$$

$$p^n: \frac{\partial v_n}{\partial t} + \beta \frac{\partial^4 v_n}{\partial x^4} = \frac{\partial^3 v_{n-1}}{\partial x^2 \partial t} + (\beta - 2)\frac{\partial^4 v_{n-1}}{\partial x^4}, \quad v_n(x,0) = 0. \tag{40}$$

The solution of (38) is

$$v_0(x,t) = \mathrm{e}^{-\beta t}\sin x. \tag{41}$$

Then (39) becomes

$$p^1: \frac{\partial v_1}{\partial t} + \beta \frac{\partial^4 v_1}{\partial x^4} = (2\beta - 2)\mathrm{e}^{-\beta t}\sin x, \quad v_1(x,0) = 0. \tag{42}$$

Now we take $\beta = 1$ so that the solution of (42) is $v_1(x,t) = 0$. In this case, $v_n(x,t) = 0$, $n \geq 2$. This in turn gives the exact solution in a closed form

$$u(x,t) = \mathrm{e}^{-t}\sin x. \tag{43}$$

In the proposed approach the rate of convergence may be accelerated, and at the same time it leads to the exact solution. To illustrate the efficiency and flexibility of the HPM, we consider the following partial differential equations.

Example 4.3 Consider the homogeneous diffusion equation that represents a heat equation with a lateral heat loss [41]:

$$u_t = u_{xx} - u, \quad 0 < x < \pi, \quad t > 0, \tag{44}$$

$$u(0,t) = 0, \quad u(\pi,t) = 0, \tag{45}$$

$$u(x,0) = \sin x. \tag{46}$$

A standard HPM: We construct the following homotopy:

$$u_t = p\left[u_{xxt} - u\right]. \tag{47}$$

Substituting (7) into (47) and equating coefficients of like powers of p, we obtain

$$p^0: \frac{\partial v_0}{\partial t} = 0, \quad v_0(x,0) = \sin x, \tag{48}$$

$$p^1: \frac{\partial v_1}{\partial t} = \frac{\partial^2 v_0}{\partial x^2} - v_0, \quad v_1(x,0) = 0, \tag{49}$$

$$\vdots$$

$$p^n: \frac{\partial v_n}{\partial t} = \frac{\partial^2 v_{n-1}}{\partial x^2} - v_{n-1}, \quad v_n(x,0) = 0. \tag{50}$$

We then obtain

$$\begin{aligned} v_0(x,t) &= \sin x, \\ v_1(x,t) &= -(2t)\sin x, \\ v_2(x,t) &= \frac{(2t)^2}{2!}\sin x, \\ v_3(x,t) &= -\frac{(2t)^3}{3!}\sin x, \\ v_4(x,t) &= \frac{(2t)^4}{4!}\sin x, \\ &\vdots \end{aligned} \tag{51}$$

from which we obtain the solution

$$u(x,t) = \sin x\left(1-(2t)+\frac{(2t)^2}{2!}-\frac{(2t)^3}{3!}+\frac{(2t)^4}{4!}-\cdots\right), \quad (52)$$

which gives the exact solution

$$u(x,t) = \mathrm{e}^{-2t}\sin x, \quad (53)$$

obtained upon using the Taylor expansion of e^{-2t}.

It should be remarked that the standard variational iteration method yields the same result as that given in (51) (see [41] for details of this).

The proposed HPM: We construct the following homotopy:

$$u_t+\beta u = p\left[u_{xx}+(\beta-1)u\right], \quad (54)$$

where β is a real number further to be determined.

Substituting (7) into (54) and equating coefficients of like powers of p, we obtain

$$p^0: \frac{\partial v_0}{\partial t}+\beta v_0 = 0, \quad v_0(x,0)=\sin x, \quad (55)$$

$$p^1: \frac{\partial v_1}{\partial t}+\beta v_1 = \frac{\partial^2 v_0}{\partial x^2}+(\beta-1)v_0, \; v_1(x,0)=0, \quad (56)$$

$$\vdots$$

$$p^n: \frac{\partial v_n}{\partial t}+\beta v_n = \frac{\partial^2 v_{n-1}}{\partial x^2}+(\beta-1)v_{n-1}, \quad v_n(x,0)=0. \quad (57)$$

The solution of (55) is

$$v_0(x,t) = \mathrm{e}^{-\beta t}\sin x. \quad (58)$$

Then (56) becomes

$$\frac{\partial v_1}{\partial t}+\beta v_1 = (\beta-2)\mathrm{e}^{-\beta t}\sin x, \quad v_1(x,0)=0. \quad (59)$$

Now we take $\beta = 2$ so that the solution of (59) is $v_1(x,t) = 0$. In this case, $v_n(x,t) = 0$, $n \geq 2$. This in turn gives the exact solution in a closed form

$$u(x,t) = \mathrm{e}^{-2t}\sin x, \quad (60)$$

which is in full agreement with that given in (53).

Example 4.4 Consider the inhomogeneous advection partial differential equation [42]

$$u_t+uu_x = x+xt^2, \quad (61)$$

$$u(x,0) = 0. \quad (62)$$

A standard HPM: We construct the following homotopy:

$$u_t-(x+xt^2) = -puu_x. \quad (63)$$

Substituting (7) into (63) and equating coefficients of like powers of p, we obtain

$$p^0: \frac{\partial v_0}{\partial t}-(x+xt^2) = 0, \quad v_0(x,0)=0, \quad (64)$$

$$p^1: \frac{\partial v_1}{\partial t} = -v_0\frac{\partial v_0}{\partial x}, \quad v_1(x,0)=0, \quad (65)$$

$$\vdots$$

$$p^n: \frac{\partial v_n}{\partial t} = -v_{n-1}\frac{\partial v_{n-1}}{\partial x}, \quad v_n(x,0)=0. \quad (66)$$

We obtain

$$v_0(x,t) = xt+\frac{1}{3}xt^3,$$
$$v_1(x,t) = -\frac{1}{3}xt^3-\frac{2}{15}xt^5-\frac{1}{63}xt^7, \quad (67)$$
$$\vdots$$

It is obvious that more components are needed to get an insight through the series solution. However, our modified approach gives the exact solution in just one iteration as in the following.

The proposed HPM: We construct the following homotopy:

$$u_t-\beta x = p\left[-uu_x+(1-\beta)x+xt^2\right], \quad (68)$$

where β is a real number further to be determined.

Substituting (7) into (68) and equating coefficients of like powers of p, we obtain

$$p^0: \frac{\partial v_0}{\partial t}-\beta x = 0, \quad v_0(x,0)=0, \quad (69)$$

$$p^1: \frac{\partial v_1}{\partial t} = -v_0\frac{\partial v_0}{\partial x}+(1-\beta)x+xt^2, \quad v_1(x,0)=0, \quad (70)$$

$$\vdots$$

The solution of (69) is

$$v_0(x,t) = \beta xt. \quad (71)$$

Then (70) becomes

$$\frac{\partial v_1}{\partial t} = (1-\beta^2)xt^2+(1-\beta)x, \quad v_1(x,0)=0. \quad (72)$$

Now we take $\beta = 1$ so that the solution of (72) is $v_1(x,t) = 0$. In this case, $v_n(x,t) = 0$, $n \geq 2$. This in turn gives the exact solution in a closed form

$$u(x,t) = xt. \quad (73)$$

5. Conclusion

In this work, we suggested a new reliable application of the homotopy perturbation method. The new application can provide a rapid convergence of the series solutions as compared with the standard HPM as well as a reliable analytic treatment of some evolution equations. Several examples were given to further reveal the power and flexibility of the HPM over existing numerical methods in handling application problems. It should be mentioned that the new application proposed in this work may require a suitable and wise choice of a homotopy equation, that should outline the basic character of the solution, so that the resulting HPM series solution lead to excellent results.

[1] J. H. He, Comput. Meth. Appl. Mech. Eng. **178**, 257 (1999).
[2] J. H. He, Int. J. Nonlinear Mech. **35**, 37 (2000).
[3] J. H. He, Appl. Math. Comput. **135**, 73 (2003).
[4] J. H. He, Appl. Math. Comput. **156**, 527 (2004).
[5] J. H. He, Appl. Math. Comput. **151**, 287 (2004).
[6] X. C. Cai, W. Y. Wu, and M. S. Li, Int. J. Nonlinear Sci. Numer. Simul. **7**, 109 (2006).
[7] A. Beléndez, C. Pascual, A. Marquez, and D. I. Méndez, Int. J. Nonlinear Sci. Numer. Simul. **8**, 483 (2007).
[8] M. L. Álvarez, D. I. Méndez, E. Fernández, M. S. Yebra, and T. Beléndez, Z. Naturforsch. **63a**, 529 (2008).
[9] J. H. He, Chaos, Solitons, and Fractals **26**, 695 (2005).
[10] J. H. He, Int. J. Nonlinear Sci. Numer. Simul. **6**, 207 (2005).
[11] J. H. He, Phys. Lett. A **347**, 228 (2005).
[12] J. H. He, Chaos, Solitons, and Fractals **26**, 827 (2005).
[13] J. H. He, Phys. Rev Lett. **90**, 174301 (2003).
[14] J. H. He, Phys. Lett. A **350**, 87 (2006).
[15] M. A. Noor and S. T. Mohyud-Din, Int. J. Nonlinear Sci. Numer. Simul. **9**, 395 (2008).
[16] J. H. He, Appl. Math. Comput. **156**, 591 (2004).
[17] M. El-Shahed, Int. J. Nonlinear Sci. Numer. Simul. **6**, 163 (2005).
[18] A. Yildirim, Comput. Math. Appl. **56**, 3175 (2008).
[19] A. Yildirim, Chaos, Solitons, and Fractals **42**, 2760 (2009).
[20] A. M. Siddiqui, R. Mahmood, and Q. K. Ghori, Int. J. Nonlinear Sci. Numer. Simul. **7**, 7 (2006).
[21] A. M. Siddiqui, R. Mahmood, and Q. K. Ghori, Int. J. Nonlinear Sci. Numer. Simul. **7**, 15 (2006).
[22] E. Yusufoglu, Int. J. Nonlinear Sci. Numer. Simul. **8**, 353 (2007).
[23] J. Biazar, M. Eslami, and H. Ghazvini, Int. J. Nonlinear Sci. Numer. Simul. **8**, 413 (2007).
[24] M. T. Darvishi and F. Khani, Z. Naturforsch. **63**, 19 (2008).
[25] A. Yildirim, Int. J. Nonlinear Sci. Numer. Simul. **9**, 111 (2008).
[26] A. Yildirim, Int. J. Comput. Math. (in press).
[27] A. Yildirim, Z. Naturforsch. **63a**, 621 (2008).
[28] E. M. E. Zayed, Z. Naturforsch. **63a**, 627 (2008).
[29] A. Yildirim, Commun. Numer. Methods Eng. (in press).
[30] T. Ozis and A. Yildirim, Int. J. Nonlinear Sci. Numer. Simul. **8**, 243 (2007).
[31] M. M. Mousa and S. F. Ragab, Z. Naturforsch. **63a**, 140 (2008).
[32] A Yildirim, Comput. Math. Appl. **57**, 612 (2009).
[33] M. Gorji, D. D. Ganji and S. Soleimani, Int. J. Nonlinear Sci. Numer. Simul. **8**, 319 (2007).
[34] J.-H. He, Non-perturbative methods for strongly nonlinear problems, dissertation.de-Verlag im Internet GmbH, Berlin 2006.
[35] J. H. He, Int. J. Mod. Phys. B **20**, 1141 (2006).
[36] J. H. He, Internat. J. Modern Phys. B **22**, 3478 (2008).
[37] J. H. He, Topol. Methods Nonlinear Anal. **31**, 205 (2008).
[38] J. H. He, Comput. Math. Appl. **57**, 410 (2009).
[39] J. H. He, Internat. J. Modern Phys. B **20**, 2561 (2006).
[40] C. Jin and M. Liu, Appl. Math. Comput. **169**, 953 (2005).
[41] A. Wazwaz, Comput. Math. Appl. **54**, 933 (2007).
[42] A. Wazwaz, Appl. Math. Comput. **102**, 77 (1999).
[43] A. Yildirim, Commun. Numer. Methods Eng. (in press).
[44] A. Yildirim, J. Math. Phys. **50**, 023510/1 (2009).
[45] A. Yildirim, Int. J. Comput. Math. (in press).