

Influence of Thermal Radiation on Blasius Flow of a Second Grade Fluid

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This work describes the series solution of two-dimensional flow and heat transfer of a second grade fluid in the presence of radiation. The governing partial differential equations are reduced into ordinary differential equations by appropriate similarity transformation. The series solutions of the resulting ordinary differential equations are obtained by using the homotopy analysis method (HAM). The convergence of the solution is discussed explicitly. The influence of pertinent parameters on the velocity and temperature is graphically displayed and discussed. Numerical values of the skin friction coefficient and the Nusselt number are also tabulated.

Key words: Blasius Flow; Heat Transfer; Series Solution.

1. Introduction

During the past few decades, the flows of non-Newtonian fluids have attracted the attention of several investigators. Such motivation infact is because of the occurrence of non-Newtonian fluids in industry and technology. To be more specific, such situations are encountered in plastic manufacture, performance of lubricants, application of paints, polymer processing, food processing, and movement of biological fluids. Many biological fluids of higher molecular weight are non-Newtonian. Geophysical applications involving ice and magma flows are highly based upon the rheological properties of non-Newtonian fluids.

It is now a well-established fact that an inadequacy of the Navier-Stokes theory has led to the development of several constitutive equations of non-Newtonian fluids. These constitutive equations contain more rheological parameters and add extra terms in the resulting equations. These resulting equations are of higher order and more complicated than the Navier-Stokes equations [1, 2]. Therefore, to develop either numerical or analytic solutions to the equations of non-Newtonian fluids is not an easy task. In view of the diversity of fluids, many constitutive equations of non-Newtonian fluids are proposed in the literature. Amongst these much attention has been focussed to the constitutive relationship in a second grade fluid. This infact is due to the simplicity of the fluid

model and the hope to obtain an analytic solution of the equations governing the flow. Some important and fundamental attempts in this direction may be mentioned through the investigations [3 – 12] and several references therein.

The radiation effect on the flow and heat transfer problem is very important from industrial point of view. Particularly, radiation effect is quite significant in nuclear power plant and satellites and space vehicles at high operating temperature. Very little is known yet about the effects of radiation on the boundary layer flow. Seddeek [13] and Raptis and Perdikis [14] studied the radiative flows of viscous fluids in the regime of magnetohydrodynamics. Very recently, Bataller [15] analysed the radiation effects in the Blasius flow.

The purpose of the present investigation is three fold. Firstly to extend the analysis of reference [15] from viscous to the second grade fluid. Secondly to include the viscous dissipation effects. Thirdly to construct an analytic solution. The structure of the paper is as follows: The mathematical formulation of the problem is described in Section 2. In Section 3, the series solutions of the velocity and the temperature are derived by using homotopy analysis method [16, 17]. This method is a very powerful analytical tool and has been already applied by many investigators to various problems [18 – 44]. Section 4 deals with the discussion of graphs and tables. Finally concluding remarks are given in Section 5.

2. Problem Formulation

We consider the two-dimensional Blasius flow and heat transfer of a second grade fluid incorporating radiation effects. Under the usual boundary layer approximations, the equations which govern the Blasius flow and heat transfer are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + \frac{\alpha_1}{\rho} \left[u \frac{\partial^3 u}{\partial y^2 \partial x} + \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} + \nu \frac{\partial^3 u}{\partial y^3} \right], \quad (2)$$

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{\partial u}{\partial y} \right)^2 + \alpha_1 \left[u \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} + v \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2} \right] - \frac{\partial q_r}{\partial y}, \quad (3)$$

subject to the following boundary conditions for the velocity and temperature:

$$u = v = 0 \text{ at } y = 0; u = U \text{ at } x = 0, \quad (4)$$

$$u \rightarrow U \text{ as } y \rightarrow \infty, \quad (5)$$

$$T = T_w \text{ at } y = 0; T = T_\infty \text{ at } x = 0, \quad (6)$$

$$T = T_\infty \text{ as } y \rightarrow \infty, \quad (7)$$

in which u , v are the velocity components, T is the temperature, ρ is the density, c_p is the specific heat, k is the thermal conductivity, μ is the dynamic viscosity, $\alpha_1 (> 0)$ is the second grade parameter, U is the free stream velocity, T_w and T_∞ are the wall temperature and the ambient fluid temperature, respectively, and q_r is the radiative heat flux which under Rosseland approximations [45] is

$$q_r = -\frac{4\sigma^*}{3k^*} \frac{\partial T^4}{\partial y}. \quad (8)$$

In the above equation σ^* and k^* denote the Stefan-Boltzmann constant and the mean absorption coefficient, respectively. We assume that the temperature differences within the flow is such as that the term T^4 may be expressed as a linear function of temperature.

Expanding T^4 in a Taylor series about T_∞ and then neglecting higher-order terms one can write

$$T^4 \cong 4T_\infty^3 T - 3T_\infty^4. \quad (9)$$

From (3), (8), and (9) follows

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \left(\alpha + \frac{16\sigma^* T_\infty^3}{3\rho c_p k^*} \right) \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p} \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\alpha_1}{\rho c_p} \left[u \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} + v \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2} \right], \quad (10)$$

where $\alpha = k/\rho c_p$ is the thermal diffusivity. Denoting the radiation parameter $N_r = kk^*/4\sigma^* T_\infty^3$, (10) can be rewritten as

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \left(\frac{3N_r + 4}{3N_r} \right) \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p} \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\alpha_1}{\rho c_p} \left[u \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} + v \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2} \right]. \quad (11)$$

If Re_x is the local Reynolds number, De is the local Deborah number, Pr is the Prandtl number and ϵ is the Brinkman number then we can define

$$\eta = y \sqrt{\frac{U}{x\nu}} = \frac{y}{x} \sqrt{Re_x}; Re_x = \frac{Ux}{\nu}, \quad (12)$$

$$u = U f'(\eta); v = \frac{1}{2} \sqrt{\frac{U\nu}{x}} (\eta f'(\eta) - f), \quad (13)$$

and introduce the following non-dimensional variables and parameters:

$$De = \frac{U\lambda}{x}, Pr = \frac{\nu}{\alpha}, \epsilon = \frac{\mu U^2}{k(T_\infty - T_w)}, \quad (14)$$

$$\theta(\eta) = \frac{T - T_w}{T_\infty - T_w}. \quad (15)$$

Equations (2) and (4) give

$$f''' + \frac{1}{2} f f'' - \frac{De}{2} [f f^{iv} + 2f' f''' - f''^2] = 0, \quad (16)$$

$$\theta'' + \frac{3N_r}{3N_r + 4} \left[\frac{Pr}{2} f \theta' + \epsilon f''^2 - \epsilon De \left\{ f f'' f''' + f' f''^2 \right\} \right] = 0. \quad (17)$$

It should be pointed out here that there is no radiation effect when $3N_r/3N_r + 4$ is equal to 1. The non-dimensional boundary conditions are

$$f = 0; \quad f' = 0 \text{ at } \eta = 0, \quad (18)$$

$$f' \rightarrow 1 \text{ as } \eta \rightarrow \infty, \quad (19)$$

$$\theta = 0 \text{ at } \eta = 0, \quad (20)$$

$$\theta \rightarrow 1 \text{ as } \eta \rightarrow \infty. \quad (21)$$

It is noticed that the skin friction coefficient C_f and the local Nusselt number Nu can be expressed as

$$C_f = \frac{\tau_w}{\rho U^2}, \quad Nu = \frac{xq_w}{k(T_\infty - T_w)}. \quad (22)$$

Here τ_w is the wall skin friction and q_w is the heat flux from the plate. The values of τ_w and q_w are

$$\tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} + \alpha_1 \left[u \frac{\partial^2 u}{\partial y \partial x} + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \nu \frac{\partial^2 u}{\partial y^2} \right] \bigg|_{y=0}, \quad (23)$$

$$q_w = -k \left. \frac{\partial T}{\partial y} \right|_{y=0}. \quad (24)$$

Invoking above values (22) gives

$$Re_x^{\frac{1}{2}} C_f = f''(0), \quad Re_x^{-\frac{1}{2}} Nu = -\theta'(0). \quad (25)$$

3. Solutions by Homotopy Analysis Method

3.1. Zeroth-Order Deformation Problems

The velocity and temperature distributions can be expressed as a set of base functions

$$\left\{ \eta^k \exp(-3n\eta) \mid k \geq 0, n \geq 0 \right\} \quad (26)$$

in the form of the following series:

$$\begin{aligned} f(\eta) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{m,n}^k \eta^k \exp(-3n\eta), \\ \theta(\eta) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} b_{m,n}^k \eta^k \exp(-3n\eta), \end{aligned} \quad (27)$$

in which $a_{m,n}^k$ and $b_{m,n}^k$ are the coefficients. Invoking the so-called rule of solution expressions for $f(\eta)$ and

$\theta(\eta)$ and (16)–(21), then the initial guesses $f_0(\eta)$ and $\theta_0(\eta)$ and the linear operators \mathcal{L}_1 and \mathcal{L}_2 are

$$f_0(\eta) = \eta + \frac{\exp(-3\eta) - 1}{3}, \quad (28)$$

$$\theta_0(\eta) = 1 - \exp(-3\eta),$$

$$\mathcal{L}_1(f) = f''' + 3f'', \quad \mathcal{L}_2(f) = f'' + 3', \quad (29)$$

where

$$\mathcal{L}_1[C_1 + C_2\eta + C_3\exp(-3\eta)] = 0, \quad (30)$$

$$\mathcal{L}_2[C_4 + C_5\exp(-3\eta)] = 0,$$

and $C_1 - C_5$ are the constants. Furthermore, (16)–(21) indicate that the nonlinear operators are

$$\begin{aligned} \mathcal{N}_1[\bar{f}(\eta, p)] &= \frac{\partial^3 \bar{f}}{\partial \eta^3} + \frac{1}{2} \frac{\partial \bar{f}}{\partial \eta} \frac{\partial^2 \bar{f}}{\partial \eta^2} \\ &\quad - \frac{De}{2} \left\{ \bar{f} \frac{\partial^4 \bar{f}}{\partial \eta^4} + 2 \frac{\partial \bar{f}}{\partial \eta} \frac{\partial^3 \bar{f}}{\partial \eta^3} - \left(\frac{\partial^2 \bar{f}}{\partial \eta^2} \right)^2 \right\}, \end{aligned} \quad (31)$$

$$\mathcal{N}_2[\bar{f}(\eta, p), \bar{\theta}(\eta, p)] = \bar{\theta}'' + \frac{3N_r}{3N_r + 4}$$

$$\cdot \left[\begin{array}{l} \frac{Pr}{2} \bar{\theta}' \bar{f} + \bar{f}''^2 \\ - \in De \{ \bar{f} \bar{f}'' \bar{f}''' + \bar{f}' \bar{f}''^2 \} \end{array} \right] = 0. \quad (32)$$

Denoting the auxiliary non-zero parameter by \hbar , the zeroth-order deformation problems can be expressed as

$$(1-p)\mathcal{L}_1[\bar{f}(\eta, p) - f_0(\eta)] = p_1 \hbar_1 \mathcal{N}_1[\bar{f}(\eta, p)], \quad (33)$$

$$(1-p)\mathcal{L}_2[\bar{\theta}(\eta, p) - \theta_0(\eta)] = p_2 \hbar_2 \mathcal{N}_2[\bar{\theta}(\eta, p)], \quad (34)$$

$$\begin{aligned} \bar{f}(0, p) &= 0, \quad \bar{f}'(0, p) = 0, \quad \bar{f}'(\infty, p) = 1, \\ \bar{\theta}(0, p) &= 0, \quad \bar{\theta}(\infty, p) = 1, \end{aligned} \quad (35)$$

where $p \in [0, 1]$ is an embedding parameter. When $p = 0$ and $p = 1$, we obtain

$$\begin{aligned} \bar{f}(\eta, 0) &= f_0(\eta), \quad \bar{f}(\eta, 1) = f(\eta), \\ \bar{\theta}(\eta, 0) &= \theta_0(\eta), \quad \bar{\theta}(\eta, 1) = \theta(\eta). \end{aligned} \quad (36)$$

Note that the initial guesses $f_0(\eta)$ and $\theta_0(\eta)$ ap-

Order of Approximations	$f''(0)$	$\theta'(0)$
5	0.31658	0.41822
10	0.37197	0.46158
15	0.37455	0.46079
20	0.37488	0.46048
25	0.37494	0.46043
30	0.37495	0.46041
35	0.37495	0.46041
40	0.37495	0.46041

Table 1. Convergence of series solution using homotopy analysis method.

De	$f''(0)$
0.0	0.33206
0.1	0.35239
0.2	0.37495
0.3	0.40000
0.4	0.42758

Table 2. Values of skin friction coefficient for different values of Deborah number De .

De	N_r	Pr	ϵ	$\theta'(0)$
0.2	1.0	1.0	2.0	0.42083
	1.3			0.46041
	1.6			0.48458
	1.3	0.5		0.41423
		1.0		0.46041
		1.5		0.49331
		1.0	0.0	0.26580
			1.0	0.36311
0.0			2.0	0.44194
0.1				0.45086
0.2				0.46041

Table 3. The values of local Nusselt number for different values of N_r , Pr , ϵ , and De .

proach $f(\eta)$ and $\theta(\eta)$, respectively, when p varies from 0 to 1. By Taylors series one can write

$$\bar{f}(\eta, p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) p^m, \quad (37)$$

$$\bar{\theta}(\eta, p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) p^m,$$

$$f_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \bar{f}(\eta, p)}{\partial p^m} \right|_{p=0}, \quad (38)$$

$$\theta_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \bar{\theta}(\eta, p)}{\partial p^m} \right|_{p=0},$$

and the convergence of the series (37) depends upon \hbar_1 and \hbar_2 . The values of \hbar_1 and \hbar_2 are chosen in such a way that the series (37) are convergent at $p = 1$. By using (36) one obtains

$$\begin{aligned} f(\eta) &= f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \\ \theta(\eta) &= \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta). \end{aligned} \quad (39)$$

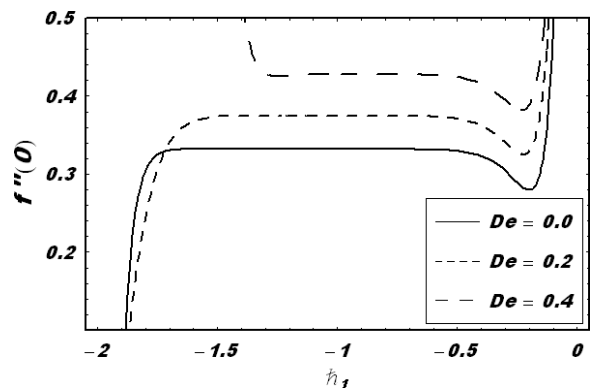


Fig. 1. \hbar -curve for the function f at 25th-order of approximation.

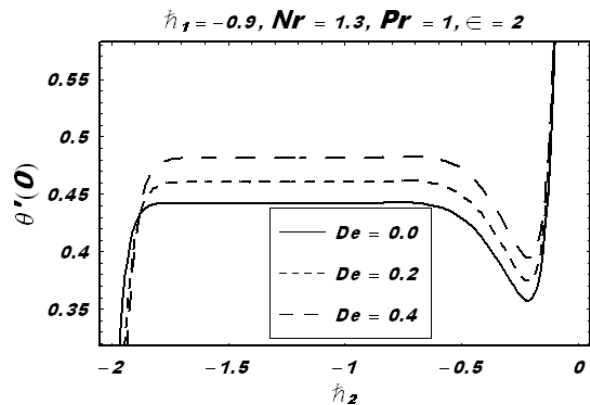


Fig. 2. \hbar -curve for the function θ at 25th-order of approximation.

3.2. m th-Order Deformation Problems

Here we first differentiate (33) and (34) m -times with respect to p and then divide by $m!$. Setting $p = 0$ we get

$$\mathcal{L}_1[f_m(\eta, p) - \chi_m f_{m-1}(\eta)] = \hbar_1 \mathcal{R}_{1,m}(\eta), \quad (40)$$

$$\mathcal{L}_2[\theta_m(\eta, p) - \chi_m \theta_{m-1}(\eta)] = \hbar_2 \mathcal{R}_{2,m}(\eta), \quad (41)$$

with the following boundary conditions:

$$\begin{aligned} f_m(0) &= f'_m(0) = f'_m(\infty) = 0, \\ \theta_m(0) &= \theta_m(\infty) = 0, \end{aligned} \quad (42)$$

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}, \quad (43)$$

$$\mathcal{R}_{1,m}(\eta) = f_{m-1}''' + \sum_{k=0}^{m-1} \left\{ \begin{array}{l} \frac{1}{2} f_{m-1-k} f_k'' \\ - \frac{De}{2} (f_{m-1-k} f_k^{iv} + 2f_{m-1-k}' f_k''' - f_{m-1-k}'' f_k'') \end{array} \right\}, \quad (44)$$

$$\mathcal{R}_{2,m}(\eta) = \theta_{m-1}'' + \frac{3N_r}{3N_r + 4} \sum_{k=0}^{m-1} \left\{ \begin{array}{l} \frac{Pr}{2} f_{m-1-k} \theta_k' + \in f_{m-1-k}'' f_k'' \\ - \in De \left(f_{m-1-k} \sum_{l=0}^k f_{k-l}'' f_l''' \right. \\ \left. + f_{m-1-k}' \sum_{l=0}^k f_{k-l}' f_l'' \right) \end{array} \right\}. \quad (45)$$

4. Analysis of the Solutions

The series given by (39) are the solutions of the present flow and heat transfer problems if one guarantees the convergence of these series. The convergence region and rate of approximations of these series solutions strongly depend upon \hbar_1 and \hbar_2 . In order to find

the admissible values of these parameters, the \hbar -curves for velocity and temperature are displayed in Figures 1 and 2. These \hbar -curves have been drawn for the physical quantities like skin friction coefficient and local Nusselt number. It is obvious from Figures 1 and 2 that the range for \hbar_1 is $-1.4 \leq \hbar_1 < 0$ and for \hbar_2 the

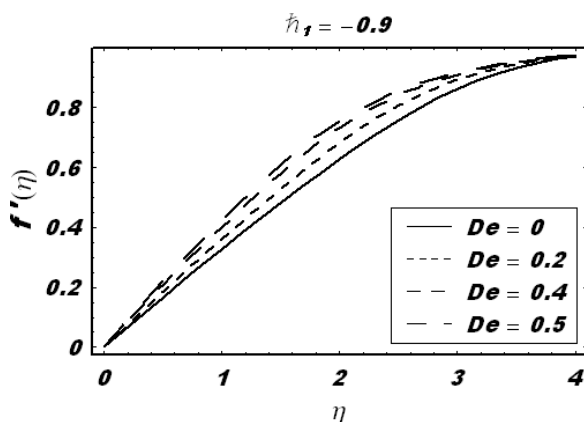


Fig. 3. Influence of Deborah number De on f' .

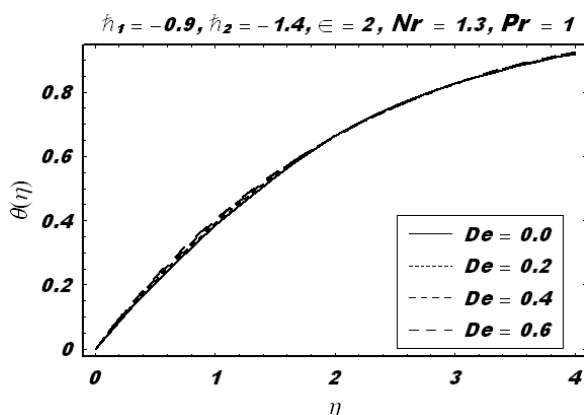


Fig. 4. Influence of Deborah number De on θ .

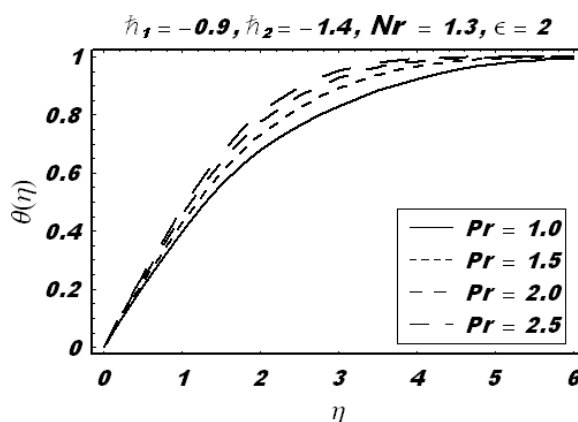


Fig. 5. Influence of Prandtl number Pr on θ .

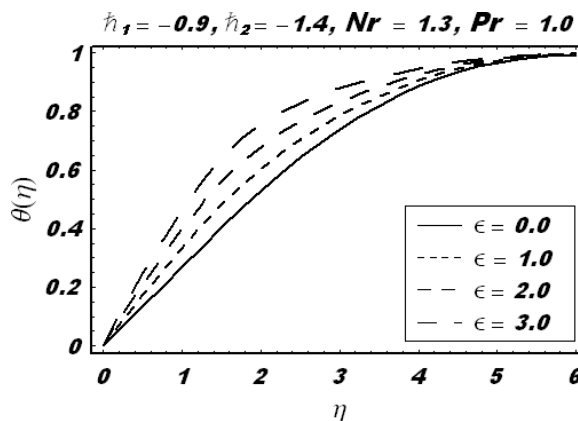


Fig. 6. Influence of Brinkman number \in on θ .

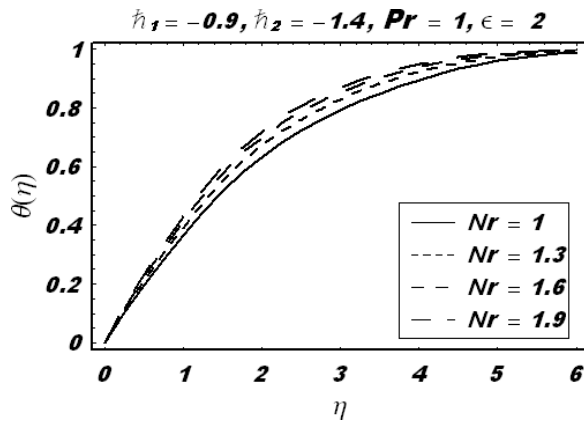


Fig. 7. Influence of radiation parameter N_r on θ .

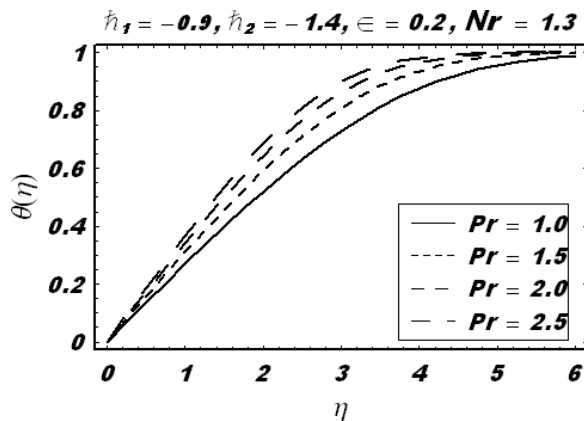


Fig. 8. Influence of Prandtl number Pr on θ . Newtonian fluid case ($De = 0$).

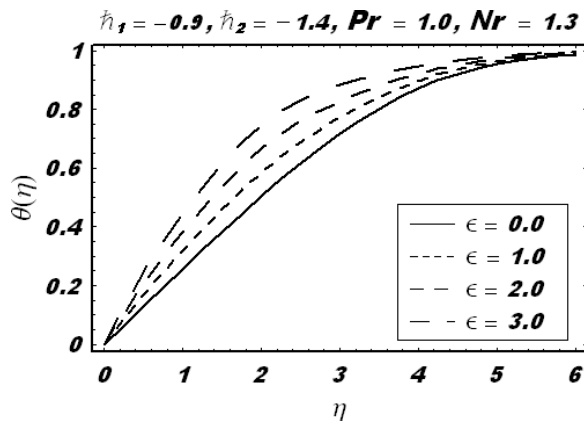


Fig. 9. Influence of Brinkman number ϵ on θ . Newtonian fluid case ($De = 0$).

range is $-1.6 \leq \bar{h}_2 < -1$. Furthermore, Figures 3–7 display the influence of Deborah number on the ve-

locity. Besides this the effect of Prandtl and Brinkman numbers on the temperature are also presented. It is evident from Figure 3 that the velocity and the boundary layer thickness increase by increasing the Deborah number. In Figure 4, the effect of Deborah number De on the temperature has been plotted. The temperature and the thermal boundary layer thickness increases by increasing the Deborah number De . Figures 5 and 6 are devoted to see the effects of Prandtl number Pr and Brinkman number ϵ on the temperature. These figures show that the temperature and the thermal boundary layer thickness increase by increasing the Prandtl and Brinkman numbers. The effects of radiation parameter N_r , are analysed by Figure 8. By increasing the radiation parameter N_r , the temperature and thermal boundary layer thickness increase when N_r , is increased.

5. Concluding Remarks

The present work provides the analytic solution for Blasius flow in a second grade fluid. Heat transfer analysis has been carried out in the presence of radiation and dissipation effects. The velocity and temperature profiles are derived. From the presented analysis the following points can be noted:

- The effects of Deborah number on the velocity and thermal boundary layer thickness is similar in a qualitative sense.
- The effects of Deborah number, Prandtl number, Brinkman number, and the radiation parameter on the thermal boundary layer thickness are similar.
- There is an increase in the skin friction coefficient when Deborah number increases.
- An excellent agreement is noted between the corresponding numerical [15] and homotopy solutions in a viscous fluid.
- The series solution in a viscous fluid which is yet not available in the literature can be obtained by choosing Deborah number equal to zero.

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