Computing the Fourier Transform via Homotopy Perturbation Method

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In this paper, the homotopy perturbation method (HPM) is applied to compute the Fourier transform (FT) of functions on \mathbb{R} . The basic properties of the Fourier transform are again obtained by HPM and examples assuring the applicability of HPM are presented.

Key words: Homotopy Perturbation Method; Fourier Transform.

1. Introduction

In recent years, the homotopy perturbation method (HPM), first proposed by He [1, 2], has successfully been applied to solve many types of linear and nonlinear functional equations. This method, which is a combination of homotopy, in topology, and classic perturbation techniques, provides a convenient way to obtain analytic or approximate solutions to a wide variety of problems arising in different fields, see [3-8] and the references therein.

In this work, we intend to use HPM for computing the Fourier transform (FT) of functions defined on \mathbb{R} . Consider the ordinary differential equation, with complex coefficients, as follows:

$$u'(x) - 2\pi i s u(x) = f(x), \tag{1}$$

where $i^2 = -1$, $s \in \mathbb{R}$, and f(x) is a real valued function on \mathbb{R} . The general solution to this equation is given by

$$u(x)e^{-2\pi isx} = \int f(x)e^{-2\pi isx}dx,$$
(2)

which leads to

$$\{u(x)e^{-2\pi isx}\}_{x=-\infty}^{\infty} = \int_{-\infty}^{\infty} f(x)e^{-2\pi isx}dx.$$
 (3)

The right hand side of the above equation is the wellknown Fourier transform of function f. So we can conclude that for computing $F(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i s x} dx$, the Fourier transform of f, we can use

$$F(s) = \{u(x)e^{-2\pi i s x}\}_{x=-\infty}^{\infty},$$
(4)

where u(x) is the solution to (1). We solve (1) via HPM and then apply (4) to obtain the Fourier transform of f.

2. HPM for Computing FT

Since the method is, by now, well-known we go straight to our subject. The interested reader can refer to He's works [1,2] for details, furthermore recent developments can be found in [9-12].

According to HPM, (1) would have a 'homotopy equation' as follows:

$$2\pi i s(\phi(x;q) - v_0(x)) + q\left(-\frac{\partial}{\partial x}\phi(x;q) + f(x) + 2\pi i s v_0(x)\right) = 0,$$
⁽⁵⁾

where v_0 is an initial guess to the solution of (1) and $q \in [0,1]$ is an embedding parameter. $\phi(x;q)$ is represented as

$$\phi(x;q) = u_0(x) + u_1(x)q + u_2(x)q^2 + \cdots$$
 (6)

When q = 0 we have $\phi(x; 0) = v_0$ and the exact solution is obtained for q = 1, i. e.

$$u(x) = \phi(x; 1) = u_0(x) + u_1(x) + u_2(x) + \cdots$$
 (7)

For obtaining this solution series, the HPM proposes to transform (5) to a system of linear equations (which are easy to solve) and then iteratively compute the u_i for $i = 1, 2, \cdots$. To do so, substituting (6) into the homotopy equation (5) and equating like powers of q we would have

$$u_0(x) = v_0(x),$$

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$$u_{1}(x) = \frac{1}{2\pi i s} (u'_{0}(x) - f(x) + 2\pi i s v_{0}(x)),$$

$$u_{2}(x) = \frac{u'_{1}(x)}{(2\pi i s)},$$

$$\vdots$$

$$u_{n}(x) = \frac{u'_{n-1}(x)}{(2\pi i s)}, n = 3, 4, \cdots.$$

Then the solution to (1) is obtained by (7). Finally, substituting this solution in expression (4), we can easily compute the FT.

Here we are free to choose the initial guess v_0 . If we set $v_0 = 0$ then the solution, according to HPM, would be of the form

$$u(x) = \sum_{n=1}^{\infty} -\frac{f^{(n-1)}(x)}{(2\pi i s)^n}.$$

So the FT is obtained by

$$F(s) = \left\{ \sum_{n=1}^{\infty} \frac{-f^{(n-1)}(x)}{(2\pi i s)^n} e^{-2\pi i s x} \right\}_{x=-\infty}^{\infty}.$$
 (8)

In the following we present some examples.

Example 1: Let $f(x) = \begin{cases} e^{-x}, & x > 0\\ 0, & x < 0 \end{cases}$, where the corresponding FT is $\frac{1}{1+2\pi i s}$. Taking $v_0(x) = 0$, as the initial energy waveled x = 1. initial guess, we would have

$$u_n(x) = \frac{(-1)^n f(x)}{(2\pi i s)^n}, \quad n = 1, 2, \cdots$$

Applying (7), the solution series has the form

$$u(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2\pi i s)^n} f(x).$$

If we restrict *s* to be $s < \frac{1}{2\pi}$, then the solution series would converge to $u(x) = \frac{-f(x)}{1+2\pi i s}$.

Employing (4) one has

$$F(s) = \left\{ \frac{-f(x)}{1+2\pi i s} e^{-2\pi i s x} \right\}_{x=-\infty}^{\infty}$$

= $\frac{-1}{1+2\pi i s} \{ f(x) e^{-2\pi i s x} \}_{x=0}^{\infty}$
= $\frac{1}{1+2\pi i s}$.

Example 2: Let $f(x) = \begin{cases} 1, & \frac{-1}{2} < x < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$, which the FT should be $\frac{\sin(\pi s)}{\pi s}$. Taking the initial guess which is the desired FT.

to be
$$v_0(x) = 0$$
, one has

$$u_1(x) = \frac{-f(x)}{2\pi i s},$$

$$u_n(x) = 0, \quad n \ge 2.$$

According to (7), the solution series has the form

$$u(x) = \frac{-f(x)}{2\pi \mathrm{i} s}.$$

Employing (4) one has

$$F(s) = \left\{ \frac{-f(x)}{2\pi i s} e^{-2\pi i s x} \right\}_{x=.}^{\infty}$$
$$= \frac{-1}{2\pi i s} \{ e^{-2\pi i s x} \}_{\frac{1}{2}}^{\frac{1}{2}}$$
$$= \frac{\sin(\pi s)}{\pi s},$$

which is the desired result.

Example 3: Let $f(x) = \begin{cases} 1 - |x|, -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$, which its FT is $\left(\frac{\sin(\pi s)}{\pi s}\right)^2$. Taking the initial guess to be $v_0(x) = 0$, we have

$$u_1(x) = \frac{-f(x)}{(2\pi i s)},$$

$$u_2(x) = \frac{-f'(x)}{(2\pi i s)^2},$$

$$u_n(x) = 0, \quad n \ge 3$$

Applying (7), the solution series has the form

$$u(x) = \frac{-f(x)}{(2\pi is)} + \frac{-f'(x)}{(2\pi is)^2}$$
$$= \frac{-1}{2\pi is} \begin{cases} 1 + x + \frac{1}{2\pi is}, & -1 < x < 0\\ 1 - x + \frac{-1}{2\pi is}, & 0 < x < 1\\ 0, & \text{otherwise} \end{cases}$$

Employing (4), one has

$$F(s) = \{u(x)e^{-2\pi isx}\}_{x=-\infty}^{\infty} = \{u(x)e^{-2\pi isx}\}_{-1}^{1}$$

= $\lim_{\varepsilon \to 0^{+}} (\{u(x)e^{-2\pi isx}\}_{-1}^{-\varepsilon} + \{u(x)e^{-2\pi isx}\}_{\varepsilon}^{1})$
= $\frac{-1}{2\pi is} \left\{\frac{2-2\cos(2\pi s)}{2\pi is}\right\}$
= $\frac{-4\sin^{2}\pi s}{(2\pi is)^{2}} = \left(\frac{\sin(\pi s)}{\pi s}\right)^{2}$,

3. Basic Properties of FT Revisited

In Fourier analysis there is no need to calculate the transform for every function, instead some fundamental rules are used to relate the new function with the transforms at hand. Here we present some well-known properties of the FT and prove them in the framework of HPM. Throughout this section f, f_1 , f_2 , and g will be functions on \mathbb{R} with the corresponding Fourier transforms F, F_1 , F_2 , and G. If we obtain g from some modification of f, there will be a corresponding modification of F that produces G. Likewise, if we obtain g from some combination of F_1 and F_2 that produces G.

3.1. Linearity

Let α be any real number, we verify that $f_1(x) + \alpha f_2(x)$ has the FT: $F_1(s) + \alpha F_2(s)$. According to (8) we have

$$\begin{split} G(s) &= \left\{ \sum_{n=1}^{\infty} \frac{-(f_1(x) + \alpha f_2(x))^{(n-1)}}{(2\pi i s)^n} e^{-2\pi i s x} \right\}_{x=-\infty}^{\infty} \\ &= \left\{ \sum_{n=1}^{\infty} \frac{-(f_1^{(n-1)}(x) + \alpha f_2^{(n-1)}(x))}{(2\pi i s)^n} e^{-2\pi i s x} \right\}_{x=-\infty}^{\infty} \\ &= \left\{ \sum_{n=1}^{\infty} \frac{-f_1^{(n-1)}(x)}{(2\pi i s)^n} e^{-2\pi i s x} \right\}_{x=-\infty}^{\infty} \\ &+ \alpha \left\{ \sum_{n=1}^{\infty} \frac{-f_2^{(n-1)}(x)}{(2\pi i s)^n} e^{-2\pi i s x} \right\}_{x=-\infty}^{\infty} \\ &= F_1(s) + \alpha F_2(s). \end{split}$$

3.2. Reflection and Conjugation

We verify the reflection rule

$$g(x) = f(-x)$$
 has the FT $G(s) = F(-s)$

by writing

$$\begin{split} G(s) &= \left\{ \sum_{n=1}^{\infty} \frac{-(f(-x))^{(n-1)}}{(2\pi i s)^n} \mathrm{e}^{-2\pi i s x} \right\}_{x=-\infty}^{\infty} \\ &= \left\{ \sum_{n=1}^{\infty} \frac{(-1)^n f^{(n-1)}(-x)}{(2\pi i s)^n} \mathrm{e}^{-2\pi i s x} \right\}_{x=-\infty}^{\infty} \\ &= \left\{ \sum_{n=1}^{\infty} \frac{f^{(n-1)}(-x)}{(2\pi i (-s))^n} \mathrm{e}^{-2\pi i (-s)(-x)} \right\}_{x=-\infty}^{\infty}, \end{split}$$

where a change of variable results in

$$G(s) = \left\{ \sum_{n=1}^{\infty} \frac{f^{(n-1)}(t)}{(2\pi i(-s))^n} e^{-2\pi i(-s)t} \right\}_{t=\infty}^{-\infty}$$
$$= \left\{ \sum_{n=1}^{\infty} \frac{-f^{(n-1)}(t)}{(2\pi i(-s))^n} e^{-2\pi i(-s)t} \right\}_{t=-\infty}^{\infty}$$
$$= F(-s).$$

The conjugation rule

$$g(x) = \overline{f(x)}$$
 has the FT $G(s) = \overline{F(-s)}$

is easily verified in the same way.

3.3. Translation and Modulation

Let x_0 be any real number, then the translation rule which is stated as

$$g(x) = f(x - x_0)$$
 has the FT $G(s) = e^{-2\pi i s x_0} F(s)$,

can be verified by writing

$$G(s) = \left\{ \sum_{n=1}^{\infty} \frac{-(f(x-x_0))^{(n-1)}}{(2\pi i s)^n} e^{-2\pi i s x} \right\}_{x=-\infty}^{\infty}$$
$$= \left\{ \sum_{n=1}^{\infty} \frac{-f^{(n-1)}(x-x_0)}{(2\pi i s)^n} e^{-2\pi i s (x-x_0)} e^{-2\pi i s x_0} \right\}_{x=-\infty}^{\infty},$$

which by a change of variable results in

$$G(s) = \left\{ \sum_{n=1}^{\infty} \frac{-f^{(n-1)}(t)}{(2\pi i s)^n} e^{-2\pi i s t} \right\}_{t=-\infty}^{\infty} e^{-2\pi i s x_0}$$

= $F(s) e^{-2\pi i s x_0}$.

Let s_0 be a real parameter, then the modulation rule

$$g(x) = e^{2\pi i s_0 x} f(x)$$
 has the FT $G(s) = F(s - s_0)$

can be verified by writing

$$G(s) = \left\{ \sum_{n=1}^{\infty} \frac{-(e^{2\pi i s_0 x} f(x))^{(n-1)}}{(2\pi i s)^n} e^{-2\pi i s x} \right\}_{x=-\infty}^{\infty}$$
$$= \left\{ \sum_{n=1}^{\infty} \frac{-1}{(2\pi i s)^n} \left[\sum_{k=0}^{n-1} \binom{n-1}{k} (2\pi i s_0)^k f^{(n-k-1)}(x) \right] \right.$$
$$\left. \cdot e^{-2\pi i (s-s_0) x} \right\}_{x=-\infty}^{\infty}.$$

Rearranging the terms according to the order of derivatives of f we have

$$\begin{split} G(s) &= \left\{ e^{-2\pi i (s-s_0)x} \left(f(x) \left\{ \frac{1}{2\pi i s} + \frac{2\pi i s_0}{(2\pi i s)^2} + \frac{(2\pi i s_0)^2}{(2\pi i s)^3} + \cdots \right\} + f'(x) \left\{ \frac{1}{(2\pi i s)^2} + \frac{2(2\pi i s_0)}{(2\pi i s)^3} + \frac{3(2\pi i s_0)^2}{(2\pi i s)^4} + \cdots \right\} + f''(x) \left\{ \frac{1}{(2\pi i s)^3} + \frac{3(2\pi i s_0)}{(2\pi i s)^4} + \frac{6(2\pi i s_0)^2}{(2\pi i s)^5} + \cdots \right\} \cdots \right) \right\}_{x=-\infty}^{\infty} \\ &= \left\{ e^{-2\pi i (s-s_0)x} \left(f(x) \left\{ \frac{1}{2\pi i (s-s_0)} \right\} \right\} \\ &+ f'(x) \left\{ \frac{1}{(2\pi i (s-s_0))^2} \right\} \\ &+ f''(x) \left\{ \frac{1}{(2\pi i (s-s_0))^3} \right\} \cdots \right) \right\}_{x=-\infty}^{\infty} \\ &= \left\{ \sum_{n=1}^{\infty} \frac{-f^{(n-1)}(x)}{(2\pi i (s-s_0))^n} e^{-2\pi i (s-s_0)x} \right\}_{x=-\infty}^{\infty} = F(s-s_0). \end{split}$$

3.4. Dilation

Let $a \neq 0$ be a real parameter. We verify the dilation rule

$$g(x) = f(ax)$$
 has the FT $G(s) = \frac{1}{|a|} F\left(\frac{s}{a}\right)$

by writing

$$G(s) = \left\{ \sum_{n=1}^{\infty} \frac{-(f(ax))^{(n-1)}}{(2\pi i s)^n} e^{-2\pi i s x} \right\}_{x=-\infty}^{\infty}$$
$$= \left\{ \sum_{n=1}^{\infty} \frac{-a^{n-1} f^{(n-1)}(ax)}{(2\pi i s)^n} e^{-2\pi i s x} \right\}_{x=-\infty}^{\infty}$$
$$= \left\{ \sum_{n=1}^{\infty} \frac{-a^{-1} f^{(n-1)}(ax)}{(2\pi i \frac{s}{a})^n} e^{-2\pi i \frac{s}{a}(ax)} \right\}_{x=-\infty}^{\infty},$$

where by a simple change of variable one has

$$G(s) = \frac{1}{|a|} \left\{ \sum_{n=1}^{\infty} \frac{-f^{(n-1)}(t)}{(2\pi i \frac{s}{a})^n} e^{-2\pi i \frac{s}{a}(t)} \right\}_{t=-\infty}^{\infty}$$
$$= \frac{1}{|a|} F\left(\frac{s}{a}\right).$$

3.5. Derivative and Power Scaling

Let f be a differentiable function. We verify the derivative rule

$$g(x) = f'(x)$$
 has the FT $G(s) = 2\pi i s F(s)$

by writing

$$G(s) = \left\{ \sum_{n=1}^{\infty} \frac{-(f'(x))^{(n-1)}}{(2\pi i s)^n} e^{-2\pi i s x} \right\}_{x=-\infty}^{\infty}$$
$$= \left\{ \sum_{n=1}^{\infty} \frac{-f^{(n)}(x)}{(2\pi i s)^n} e^{-2\pi i s x} \right\}_{x=-\infty}^{\infty},$$

if we add and subtract f(x), then the above can be written as follows:

$$G(S) = \left\{ \left(f(x) + 2\pi is \left\{ -\frac{f(x)}{2\pi is} - \frac{f'(x)}{(2\pi is)^2} - \frac{f''(x)}{(2\pi is)^3} + \cdots \right\} \right) e^{-2\pi isx} \right\}_{x=-\infty}^{\infty}$$
$$= \left\{ \left(f(x) + 2\pi is \sum_{n=0}^{\infty} \frac{-f^{(n)}(x)}{(2\pi is)^{n+1}} \right) e^{-2\pi isx} \right\}_{x=-\infty}^{\infty}$$
$$= \left\{ f(x) e^{-2\pi isx} \right\}_{x=-\infty}^{\infty}$$
$$+ 2\pi is \left\{ \sum_{n=0}^{\infty} \frac{-f^{(n)}(x)}{(2\pi is)^{n+1}} e^{-2\pi isx} \right\}_{x=-\infty}^{\infty},$$

where the first expression is zero because we suppose that f has a Fourier transform (so it should vanish at $\pm\infty$), so it leads to

$$G(s) = 2\pi i s F(s).$$

To establish the power scaling rule

$$g(x) = xf(x)$$
 has the FT $G(s) = \frac{-1}{2\pi i}F'(s)$

we write

$$G(s) = \left\{ \sum_{n=1}^{\infty} \frac{-(xf(x))^{(n-1)}}{(2\pi i s)^n} e^{-2\pi i s x} \right\}_{x=-\infty}^{\infty} = \left\{ \left(\frac{-xf(x)}{(2\pi i s)} + \sum_{n=2}^{\infty} \frac{-1}{(2\pi i s)^n} \left\{ xf^{(n-1)}(x) + (n-1) + (n-1) \right\} + f^{(n-2)}(x) \right\} \right\}_{x=-\infty}^{\infty} = \left\{ \left(-\sum_{n=2}^{\infty} (n-1) + \frac{f^{(n-2)}(x)}{(2\pi i s)^n} - \sum_{n=1}^{\infty} \frac{xf^{(n-1)}(x)}{(2\pi i s)^n} \right) e^{-2\pi i s x} \right\}_{x=-\infty}^{\infty}$$

and a change in the index results in

$$G(s) = \left\{ \sum_{n=1}^{\infty} \left(-\frac{nf^{(n-1)}(x)}{(2\pi i s)^{n+1}} - \frac{xf^{(n-1)}(x)}{(2\pi i s)^n} \right) e^{-2\pi i s x} \right\}_{x=-\infty}^{\infty}.$$

The latter expression can be written as follows:

$$G(s) = \frac{-1}{2\pi i} \left\{ \sum_{n=1}^{\infty} -\left(\frac{n f^{(n-1)}(x)(-2\pi i)}{(2\pi i s)^{n+1}} + \frac{(-2\pi i x) f^{(n-1)}(x)}{(2\pi i s)^n} \right) e^{-2\pi i s x} \right\}_{x=-\infty}^{\infty} = \frac{-1}{2\pi i} F'(s) e^{-2\pi i s x} \left\{ \sum_{n=1}^{\infty} -\frac{(-2\pi i x) f^{(n-1)}(x)}{(2\pi i s)^{n+1}} + \frac{(-2\pi i x) f^{(n-1)}(x)}{(2\pi i s)^n} \right\}_{x=-\infty}^{\infty} = \frac{-1}{2\pi i} e^{-2\pi i s x} \left\{ \sum_{n=1}^{\infty} -\frac{(-2\pi i x) f^{(n-1)}(x)}{(2\pi i s)^{n+1}} + \frac{(-2\pi i x) f^{(n-1)}(x)}{(2\pi i s)^n} \right\}_{x=-\infty}^{\infty} = \frac{-1}{2\pi i s} e^{-2\pi i s x} \left\{ \sum_{n=1}^{\infty} -\frac{(-2\pi i x) f^{(n-1)}(x)}{(2\pi i s)^{n+1}} + \frac{(-2\pi i x) f^{(n-1)}(x)}{(2\pi i s)^n} \right\}_{x=-\infty}^{\infty} = \frac{-1}{2\pi i s} e^{-2\pi i s x} \left\{ \sum_{n=1}^{\infty} -\frac{(-2\pi i x) f^{(n-1)}(x)}{(2\pi i s)^{n+1}} + \frac{(-2\pi i x) f^{(n-1)}(x)}{(2\pi i s)^n} \right\}_{x=-\infty}^{\infty} = \frac{-1}{2\pi i s} e^{-2\pi i s x} e^{-2\pi$$

4. Conclusion and Suggestions

The homotopy perturbation method is an elegant method which is easy to use. Its applicability in computing integral transforms such as Laplace transform [13] and Sumudu Transform [14] have been proved by scientists. In this paper, we have verified its efficiency in computing the Fourier transform, which is the most important integral transform. The classic calculation of Fourier transforms involves a computation of an infinite range definite integral. Instead, the proposed method based on HPM uses differentiations, so it can be used as an alternative. We have also ver-

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ified the basic properties of the Fourier transform in this new frame work. Using these properties and the presented examples, it would be easy to calculate the Fourier transform of a large number of functions.

It would be a good idea to use HPM to evaluate more integral transforms and specially twodimensional cases of the mentioned ones.

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