Thin Film Flow of a Third Grade Fluid with Variable Viscosity

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The effects of variable viscosity on the flow and heat transfer in a thin film flow for a third grade fluid has been discussed. The thin film is considered on the outer side of an infinitely long vertical cylinder. The governing nonlinear differential equations of momentum and energy are solved analytically by using homotopy analysis method. The expression for the viscous dissipation and entropy generation are also defined. The graphical results are presented for various physical parameters appearing in the problem.

Key words: Third Grade Fluid; Variable Viscosity; Thin Film Flow; Constant Pressure Gradient.

1. Introduction

Various applications of heat transfer inside thin films in several industrial manufacturing processes have led to renewed interest among the researchers. Some of the typical applications of such flows are wire and fiber coating, reactor fluidization, polymer processing, food stuff processing transpiration cooling, microchip production, and lining of mammalian lungs. Several authors have considered flow inside a thin film like Siddiqui et al. [1-3], Sajid and Hayat [4] and Hayat and Sajid [5] have discussed the thin film flows of non-Newtonian fluids.

One of the important aspects in this theoretical study is the investigation of non-Newtonian fluid with variable viscosity. This is due to the fact that the typical Navier-Stokes theory becomes insufficient for the description of some complex rheological fluids such as shampoo, blood, paints, polymer solutions, and plastic films. In view of this, Massoudi and Christie [6] investigated the effects of variable viscosity and the description on the flow of a third grade fluid in a pipe. Using this idea Pakdermirli and Yilbas [7-8] have presented the analytic solution by using perturbation technique and also found the entropy generation number for both constants and Vogel's model of viscosities.

In all these above mentioned investigations [1-8], the thin film flow with variable viscosity has not been taken into account. In the present study the thin film flow of a third grade fluid with variable viscosity in the presence of a constant pressure gradient is discussed. An analytic solution is presented using homotopy analysis method, a brief overview of the method is given in references [9-23]. At the end the analytical solutions are discussed graphically. To the best of the author's knowledge the thin film flow of third grade fluid with variable viscosity has not been reported in the literature until now.

2. Mathematical Formulation

Let us consider an incompressible, thermodynamically compatible, steady third grade fluid with variable viscosity, lying on the outer surface of an infinitely long vertical cylinder. The flow is in the form of thin, uniform axially symmetric film of thickness δ , in axially contact with the stationary air. The governing equation of motion and energy in non-dimensional form for cylindrical coordinates are [6, 7]

$$\frac{\mathrm{d}\mu}{\mathrm{d}r}\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{\mu}{r}\left[\frac{\mathrm{d}v}{\mathrm{d}r} + r\frac{\mathrm{d}^2v}{\mathrm{d}r^2}\right] + \frac{\Lambda}{r}\left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^2 \left[\frac{\mathrm{d}v}{\mathrm{d}r} + 3r\frac{\mathrm{d}^2v}{\mathrm{d}r^2}\right] + \frac{K}{\mu}r = C,$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\theta}{\mathrm{d}r} + \Gamma\left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^2 \left[\mu + \Lambda\left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^2\right] = 0. (2)$$

The corresponding boundary conditions for the thin film flow are

$$v(r) = 0, \quad \theta(r) = 0 \text{ at } r = 1,$$
 (3)

$$\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{\mathrm{d}\theta}{\mathrm{d}r} = 0 \text{ at } r = d.$$
(4)

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The first condition being the no-slip condition at r = 1and the second comes from the stress tensor τ_{r_z} . In the above equations the non-dimensionalize variables are defined as

$$r = \frac{\bar{r}}{R}, \quad \Gamma = \frac{\mu_* V_0^2}{k(\theta_m - \theta_w)}, \quad \Lambda = \frac{2(\beta_2 + \beta_3)V_0^2}{R^2 \mu_*}, \quad (5)$$
$$C_1 = \frac{\partial p}{\partial z}, \quad C = \frac{C_1 R^2}{\mu_* V_0}, \quad d = 1 + \frac{\delta}{R}, \quad K = \frac{\rho_g R^2}{\mu_* V_0},$$

where \bar{r} , μ_* , θ_m , θ_w , V_0 , Λ , β_2 , β_3 , R, δ are defined in [7] and g is the constant of gravity.

The Reynold model of viscosity is used to describe the variation of viscosity with temperature. The nondimensional variable of third grade fluid Λ may also be dependent on the temperature, but for the sake of simplicity here Λ is treating as constant. The Reynold model of viscosity is defined as

$$\mu = e^{-M\theta}.$$
 (6)

Using the Maclaurin series expansion the above expression can be written as

$$\mu = 1 - M\theta + O(\theta^2). \tag{7}$$

Here M = 0 corresponds to the constant viscosity case. Using (7) in (1) and (2), we have

$$-M\frac{\mathrm{d}\theta}{\mathrm{d}r}\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{1}{r}\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{M}{r}\theta\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{\mathrm{d}^{2}v}{\mathrm{d}r^{2}} - M\theta\frac{\mathrm{d}^{2}v}{\mathrm{d}r^{2}} + \frac{\Lambda}{r}\left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{3} + 3\Lambda\frac{\mathrm{d}^{2}v}{\mathrm{d}r^{2}}\left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{2}$$
(8)
$$+K(1+M\theta) = C,
$$\frac{\mathrm{d}^{2}\theta}{\mathrm{d}r^{2}} + \frac{1}{r}\frac{\mathrm{d}\theta}{\mathrm{d}r} + \Gamma\left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{2} - \Gamma M\theta\left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{2} + \Lambda\Gamma\left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{4} = 0.$$
(9)$$

3. Solution by Homotopy Analysis Method

The equations (8) and (9) are the governing nonlinear ordinary differential equation having boundary conditions given by (3) and (4). We are going to solve the above boundary value problem with the help of the homotopy analysis method [9-17]. Let

$$v_0(r) = \frac{C}{2}(r^2 - 1) - Cd(r - 1), \tag{10}$$

$$\theta_0(r) = \frac{C^2 \Gamma}{12} ((1-d)^4 - (r-d)^4)$$
(11)

denote the initial guess (for velocity and temperature). The auxiliary linear operators for velocity and temperature are

$$\mathcal{L}_{\nu}(\nu) = \nu'' + \frac{1}{r}\nu',$$
(12)

$$\mathcal{L}_{\theta}(\theta) = \theta'' + \frac{1}{r}\theta', \qquad (13)$$

as the auxiliary linear operators satisfying

$$\mathcal{L}_{vr}(A_1 + B_1 \ln r) = 0, \tag{14}$$

$$\mathcal{L}_{\theta r}(A_2 + B_2 \ln r) = 0, \tag{15}$$

where A_1, A_2, B_1, B_2 are constants.

The zeroth-order deformation problem is written as

$$(1-p)\mathcal{L}_{\nu}[\bar{\nu}(r,p)-\nu_{0}(r)] = p\hbar_{\nu}N_{\nu}[\bar{\nu}(r,p),\bar{\theta}(r,p)],$$
(16)

$$\begin{aligned} &(1-p)\mathcal{L}_{\theta}[\;\bar{\theta}(r,p)-\theta_{0}\;(r)]\\ &=p\hbar_{\theta}N_{\theta}[\;\bar{v}\;(r,p),\bar{\theta}(r,p)], \end{aligned}$$
(17)

$$\bar{v}(r,p) = \bar{\theta}(r,p) = 0 \text{ for } r = 1, \qquad (18)$$

$$\frac{\partial \bar{v}(r,p)}{\partial r} = \frac{\partial \theta(r,p)}{\partial r} = 0 \text{ for } r = d, \qquad (19)$$

$$N_{\nu}\left[\bar{v}\left(r,p\right),\bar{\theta}\left(r,p\right)\right]$$

$$= -M\frac{\mathrm{d}\theta}{\mathrm{d}r}\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{1}{r}\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{M}{r}\theta\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{\mathrm{d}^{2}v}{\mathrm{d}r^{2}}$$

$$-M\theta\frac{\mathrm{d}^{2}v}{\mathrm{d}r^{2}} + \frac{\Lambda}{r}\left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{3} + 3\Lambda\frac{\mathrm{d}^{2}v}{\mathrm{d}r^{2}}\left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{2} \quad (20)$$

$$+K(1+M\theta)r - C,$$

$$N_{\theta}\left[\bar{v}\left(r,p\right),\bar{\theta}\left(r,p\right)\right]$$

$$= \frac{\mathrm{d}^{2}\theta}{\mathrm{d}r^{2}} + \frac{1}{r}\frac{\mathrm{d}\theta}{\mathrm{d}r} + \Gamma\left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{2} \quad (21)$$

$$-\Gamma M\theta\left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{2} + \Lambda\Gamma\left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{4},$$

where $p \varepsilon[0, 1]$ is the embedding parameter and \hbar_v and \hbar_{θ} are auxiliary non-zero operators.

The mth-order deformation equations are

$$\mathcal{L}_{\nu}[\nu_m(r) - \boldsymbol{\chi}_m \nu_{m-1}(r)] = \hbar_{\nu} R_{\nu}(r), \qquad (22)$$

$$\mathcal{L}_{\theta}[\theta_m(r) - \chi_m \theta_{m-1}(r)] = \hbar_{\theta} R_{\theta}(r), \qquad (23)$$



$$R_{v} = -M \sum_{k=0}^{m-1} v'_{m-1-k} \theta'_{k} + \frac{1}{r} v'_{m-1} - \frac{M}{r} \sum_{k=0}^{m-1} v'_{m-1-k} \theta_{k}$$

+ $v''_{m-1} - M \sum_{k=0}^{m-1} v''_{m-1} \theta_{k} + \frac{\Lambda}{r} \sum_{k=0}^{m-1} v'_{m-1-k} \sum_{l=0}^{k} v'_{k-l} v'_{l}$
+ $KMr\theta_{m-1} + 3\Lambda \sum_{k=0}^{m-1} v'_{m-1-k} \sum_{l=0}^{k} v'_{k-l} v''_{l}$
+ $(Kr - C)(1 - \chi_{m}),$ (24)

$$R_{\theta} = \frac{1}{r} \theta'_{m-1} + \theta''_{m-1} + \Gamma \sum_{k=0}^{m-1} v'_{m-1-k} v'_{k} + \Gamma M \sum_{k=0}^{m-1} v'_{m-1-k} \sum_{l=0}^{k} v'_{k-l} \theta_{l} - \Lambda \Gamma \sum_{k=0}^{m-1} v'_{m-1-k} \sum_{k=0}^{m-1} v'_{k-l} \sum_{s=0}^{k} v'_{l-s} v'_{s}.$$
(25)

We now use the symbolic calculation software MATH-EMATICA and solve the set of linear differential equation (34) and (35) with conditions up to the first few orders of approximations. It is found that $v_m(r)$ can be written as

$$v_m(r) = \sum_{n=1}^{\infty} a_{m,n} r^{3n+4}, \quad m \ge 0,$$

$$\theta_m(r) = \sum_{n=1}^{\infty} b_{m,n} r^{5n+4}, \quad m \ge 0,$$
(26)

where $a_{m,n}$ and $b_{m,n}$ are constants which can be determined on substituting (26) into (22) and (23).

Fig. 1. h-curve for different values of variable viscosity parameter.

4. Viscous Dissipation and Entropy Generation

The non-dimensional viscous dissipation and entropy generation can be defined as in [7]

$$\bar{\phi} = \frac{\mu_* V_0^2}{R^2} \left(\frac{\mathrm{d}\nu}{\mathrm{d}r}\right)^2 \left[\mu + \Lambda \left(\frac{\mathrm{d}\nu}{\mathrm{d}r}\right)^2\right].$$
 (27)

The non-dimensional entropy generation is defined as

$$N_{S} = \left(\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)^{2} + \Gamma \theta_{0} \left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{2} \left[\mu + \Lambda \left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{2}\right], \quad (28)$$

where

$$N_{S} = \frac{S_{gen}^{'''}}{S_{G}^{'''}}, \quad S_{G}^{'''} = \frac{k(\theta_{m} - \theta_{w})^{2}}{R^{2}\bar{\theta}_{0}^{2}}.$$
 (29)

The first term in (28) is due to the heat generation and the second term is due to the viscous dissipation. We split N_s into two parts such as

$$N_S = N_{S1} + N_{S2}, (30)$$

where

$$N_{S1} = \left(\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)^2,\tag{31}$$

$$N_{S2} = \Gamma \theta_0 \left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^2 \left[\mu + \Lambda \left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^2\right].$$
 (32)

With the help of (26) into (27)-(32), these physical numbers can be easily calculated, which are obvious results.



Fig. 2. Velocity profile along the radial distance for different values of M.



Fig. 3. Temperature profile along the radial distance for different values of M.

5. Graphical Results and Discussion

In this paper, I have presented a third grade fluid lying on the outer surface of an infinitely long cylinder. The Reynold model is accounted for temperature dependent viscosity.

Figure 1 is prepared to see the convergence region for different values of M. It is observed from the figure that the solution is convergent when $-0.3 \le h \le$ -0.05. It is also observed that as we increase the value of M the convergence region becomes smaller, the similar effects are seen for the convergence region of Λ , but that graph is not shown here to avoid the repetition. In Figure 2 is plotted the velocity distribution Vagainst r for different values of M when the film thickness is 2. It is seen from the figure that with the increase in M, the velocity is increasing and a narrow film is seen. Figure 3 is prepared for the temperature distribution. The figure shows that with the increase in M the temperature decreases. This happens



Fig. 4. Velocity profile along the radial distance for different values of Λ .



Fig. 5. Temperature profile along the radial distance for different values of Λ .



Fig. 6. Velocity profile along the radial distance for different values of Γ .

because when we increase M the viscosity decreases and decrease of viscosity effects the viscous dissipation which causes the decrease in temperature. The effects of Λ on velocity and temperature are shown in Figures 4 and 5. In this case, with the increase in Λ



Fig. 7. Temperature profile along the radial distance for different values of Γ .



Fig. 8. Velocity profile along the radial distance for different values of *C*.

the velocity decreases but the temperature increases. The effects of Brikmann number Γ are shown in Figures 6 and 7. It is found that with the increase in Γ , both velocity and temperature increases. The effects of constant pressure gradient on the velocity and temperature can be seen in Figures 8 and 9. It is observed from the figures that with the increase in *C*, both the

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Fig. 9. Temperature profile along the radial distance for different values of *C*.



Fig. 10. Velocity profile along the radial distance for different values of *K*.

velocity and temperature increases and gives the maximum value at the free surface. The velocity field for different values of gravitation associated constant K is illustrated in Figure 10. It is observed that the velocity field increases with the increase in K and the behaviour of the velocity is almost similar to that of pressure drop C.

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