Non-Travelling Wave Solutions of the (2+1)-Dimensional Dispersive Long Wave System

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With the aid of symbolic computation and the extended F-expansion method, we construct more general types of exact non-travelling wave solutions of the (2+1)-dimensional dispersive long wave system. These solutions include single and combined Jacobi elliptic function solutions, rational solutions, hyperbolic function solutions, and trigonometric function solutions.

Key words: Extended F-Expansion Method; Exact Solutions; (2+1)-Dimensional Dispersive Long Wave System; Soliton-Like Solution; Jacobi Elliptic Function Solutions.

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1. Introduction

It is important to seek for more explicit exact solutions of nonlinear partial differential equations (NLPDEs) in mathematical physics. With the help of symbolic computation software like Maple or Mathematica, much work has been focused on the various extensions and applications of the known methods to construct exact solutions of NLPDEs. Mathematical modelling of physical systems often leads to nonlinear evolution equations (NLEEs). The study of (2+1)-dimensional NLEEs, or even higher dimensional NLEEs, has also attracted more attention. There are many powerful and direct methods to construct the exact solutions of NLPDEs, such as the inverse scattering transform [1], tanh-function method [2 – 4], the generalized hyperbolic function method [5, 6], Exp-function method [7], sine/cosine method [8] and so on. Recently, many exact solutions expressed by Jacobi elliptic functions (JEFs) of NLEEs have been obtained by Jacobi elliptic function expansion method [9 – 11], mapping method [12, 13], F-expansion method [14], the extended F-expansion method [15], the improved generalized F-expansion method [16, 17], the generalized Jacobi elliptic function method [18, 19], the variable-coefficient F-expansion method [20] and other methods [21 – 23]. The F-expansion method [14] is an over-all generalization of Jacobi elliptic function expansion method. Using many methods [3 – 6, 9 – 11, 14, 15, 22, 23], we can get only the travelling wave solutions. Ren and Zhang [16] and Zhang and Xia [17] improved some methods to seek for more types of non-travelling wave and coefficient function solutions.

In this paper, we consider the (2+1)-dimensional dispersive long wave (DLW) system

$$\begin{aligned} &u_{yt} + v_{xx} + u_x u_y + u u_{xy} = 0, \\ &v_t + u_x + v u_x + u v_x + u_{xxy} = 0. \end{aligned} \tag{1}$$

System (1) was first obtained by Boiti et al. [24] and possesses a Kac-Moody-Virasoro structure [25]. Searching for methods to find more types of exact solutions of the system (1) is of fundamental interest in fluid dynamics. Some soliton-like solutions of the (2+1)-dimensional DLW system (1) were obtained in [26, 27]. Recently, some exact solutions of (1) were constructed by using various methods, including travelling-wave solutions, soliton-like solutions, periodic wave solutions and Weierstrass function solutions [28 – 33]. A good understanding of the solutions for system (1) is very helpful to coastal and civil engineers in applying the nonlinear water model to coastal harbor design. The present work is motivated by the desire to improve the work made in [28, 29] by proposing a more general ansatz solution, in which the restriction on coefficients being constants are removed, to obtain some new and more general exact solutions of (1) by using the extended F-expansion method with the help of symbolic computation. More recently, Zhang et al. [34] and Bai and Niu [35] derived many types of exact solutions of system (1) by using different methods.

This paper is organized as follows: in Section 2, we describe the variable-coefficient extended F-expansion method [15, 20] to construct exact solutions for a system of NLPDEs. In Section 3, we apply this method to the (2+1)-dimensional DLW system (1). We draw plots to show the properties of some JEF solutions. In Section 4, we conclude the paper.

2. The Variable-Coefficient Extended F-Expansion Method

Let us simply describe the variable-coefficient extended F-expansion method, as follows:

Step 1. Consider a given system of NLPDEs with independent variables $x = (x_0 \equiv t, x_1, x_2, \ldots, x_n)$ and dependent variables u and v in the form

$$\begin{aligned} &P(u, v, u_t, u_{x_i}, v_{x_i}, u_{x_i x_j}, v_{x_i x_j}, \ldots) = 0, \\ &Q(u, v, u_t, u_{x_i}, v_{x_i}, u_{x_i x_j}, v_{x_i x_j}, \ldots) = 0, \end{aligned} \tag{2}$$

where the functions P and Q are polynomial functions of u, v and their derivatives, and the subscripts denote partial derivatives. We seek the solutions in the form

$$\begin{aligned} u(x) &= \sum_{i=0}^{n} (A_i(x) F^i(\xi) + B_i(x) F^{-i}(\xi)), \\ v(x) &= \sum_{j=0}^{M} (a_j(x) F^j(\xi) + b_j(x) F^{-j}(\xi)), \end{aligned} \tag{3}$$

where $A_0(x)$, $A_i(x)$, $B_i(x)$, $(i = 1, \ldots, n)$, $a_0(x)$, $a_j(x)$, $b_j(x)$ $(j = 1, \ldots, M)$ and $\xi(x)$ are all differentiable functions of x to be determined and $F(\xi)$ satisfies the first order nonlinear ordinary differential equation (ODE)

$$(F'(\xi))^2 = q_0 + q_2 F^2(\xi) + q_4 F^4(\xi), \tag{4}$$

where q_0, q_2 and q_4 are constants and the prime denotes $\mathrm{d}/\mathrm{d}\xi$.

Step 2. Determine the integer numbers n and M by balancing the highest-order derivative terms with the nonlinear terms in system (2).

Step 3. Substituting (3) with (4) into (2), then the left-hand side of system (2) can be converted into a polynomial in $F(\xi)$. Setting each coefficient to zero to derive a system of PDEs for $A_0(x)$, $A_i(x)$, $B_i(x)$, $a_0(x)$, $a_j(x)$, $b_j(x)$, and $\xi(x)$.

Table 1. The ordinary differential equation and Jacobi elliptic functions. Relation between values of (q_0, q_2, q_4) and corresponding $F(\xi)$ in ODE

$$(F')^2 = q_0 + q_2 F^2 + q_4 F^4.$$

q_0	q_2	q_4	F
1	$-1-m^2$	m^2	$\mathrm{sn}\xi$, $\mathrm{cd}\xi = \frac{\mathrm{cn}\xi}{\mathrm{dn}\xi}$
$1-m^2$	$2m^2-1$	$-m^2$	$\mathrm{cn}\xi$
m^2-1	$2-m^2$	-1	$\mathrm{dn}\xi$
m^2	$-1-m^2$	1	$\mathrm{ns}\xi = \frac{1}{\mathrm{sn}\xi}$, $\mathrm{dc}\xi = \frac{\mathrm{dn}\xi}{\mathrm{cn}\xi}$
$-m^2$	$2m^2-1$	$1-m^2$	$\mathrm{nc}\xi = \frac{1}{\mathrm{cn}\xi}$
-1	$2-m^2$	m^2-1	$\mathrm{nd}\xi = \frac{1}{\mathrm{dn}\xi}$
1	$2-m^2$	$1-m^2$	$\mathrm{sc}\xi = \frac{\mathrm{sn}\xi}{\mathrm{cn}\xi}$
1	$2m^2-1$	$-m^2(1-m^2)$	$\mathrm{sd}\xi = \frac{\mathrm{sn}\xi}{\mathrm{dn}\xi}$
$1-m^2$	$2-m^2$	1	$\mathrm{cs}\xi = \frac{\mathrm{cn}\xi}{\mathrm{sn}\xi}$
$-m^2(1-m^2)$	$2m^2-1$	1	$\mathrm{ds}\xi = \frac{\mathrm{dn}\xi}{\mathrm{sn}\xi}$
$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^4}{4}$	$\frac{\mathrm{sn}\xi}{1 \pm \mathrm{dn}\xi}$
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\mathrm{sn}\xi \pm \mathrm{i}\,\mathrm{cn}\xi$

Step 4. Solving the system obtained in the above step by the use of Maple. In order to make the calculation feasible, we may choose some special forms of $A_i(x)$, $B_i(x)$, $a_j(x)$, $b_j(x)$, and $\xi(x)$.

Step 5. Substituting these results into (3) to derive various solutions of (2) depending on the solutions of equation (4). Many kinds of JEF solutions of equation (4) are listed in Table 1.

The JEFs $\mathrm{sn}\xi = \mathrm{sn}(\xi, m)$, $\mathrm{cn}\xi = \mathrm{cn}(\xi, m)$, and $\mathrm{dn}\xi = \mathrm{dn}(\xi, m)$, where m $(0 < m < 1)$ is the modulus of the elliptic function, are double periodic and posses the following properties:

$$\mathrm{sn}^2\xi + \mathrm{cn}^2\xi = 1, \quad \mathrm{dn}^2\xi + m^2\mathrm{sn}^2\xi = 1,$$
$$\frac{\mathrm{d}}{\mathrm{d}\xi}(\mathrm{sn}\xi) = \mathrm{cn}\xi\,\mathrm{dn}\xi, \quad \frac{\mathrm{d}}{\mathrm{d}\xi}(\mathrm{cn}\xi) = -\mathrm{sn}\xi\,\mathrm{dn}\xi,$$
$$\frac{\mathrm{d}}{\mathrm{d}\xi}(\mathrm{dn}\xi) = -m^2\mathrm{sn}\xi\,\mathrm{cn}\xi.$$

The JEFs degenerate into hyperbolic functions when $m \to 1$:

$$\mathrm{sn}\xi \to \tanh\xi, \quad \mathrm{cn}\xi \to \mathrm{sech}\xi, \quad \mathrm{dn}\xi \to \mathrm{sech}\xi,$$

while the JEFs degenerate into trigonometric functions when $m \to 0$:

$$\mathrm{sn}\xi \to \sin\xi, \quad \mathrm{cn}\xi \to \cos\xi, \quad \mathrm{dn}\xi \to 1.$$

Some more properties of JEFs can be found in [36], notation see Table 1.

3. Exact JEF Solutions of the (2+1)-Dimensional DLW System

In this section, we obtain new exact JEF solutions of the (2+1)-dimensional DLW system by using the variable-coefficient extended F-expansion method. Balancing the highest-order derivative term with the nonlinear term in system (1) gives $n = 1$ and $M = 2$, so we suppose that equation (1) has the following formal solution:

$$\begin{aligned} u(x,y,t) &= A_0 + A_1 F(\xi) + B_1 F^{-1}(\xi), \\ v(x,y,t) &= a_0 + a_1 F(\xi) + a_2 F^2(\xi) \\ &\quad + b_1 F^{-1}(\xi) + b_2 F^{-2}(\xi), \end{aligned} \tag{5}$$

where $A_0(y,t) \equiv A_0$, $A_1(y,t) \equiv A_1$, $B_1(y,t) \equiv B_1$, $a_0(y,t) \equiv a_0$, $a_1(y,t) \equiv a_1$, $a_2(y,t) \equiv a_2$, $b_1(y,t) \equiv b_1$, $b_2(y,t) \equiv b_2$, and $\xi = xk(y,t) + \eta(y,t)$. Substituting (5) into (1), and equating each of the coefficients of $F(\xi)$ to zero, we get a set of PDEs for A_0, A_1, B_1, a_0, a_1, a_2, b_1, b_2 and ξ. Solving the set of PDEs by the use of Maple, we have

Case 1:

$$\begin{aligned} &B_1 = a_1 = b_1 = b_2 = 0, \quad k(y,t) = C_1, \\ &A_0(y,t) = -\frac{1}{C_1}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t}, \quad A_1 = \pm 2\sqrt{q_4}C_1, \\ &\eta(y,t) = -\int \frac{(1+f_1(y))\mathrm{d}y}{q_2 C_1} + f_3(t), \\ &a_0 = f_1(y), \quad a_2 = 2q_4(1+f_1(y))/q_2. \end{aligned} \tag{6}$$

Case 2:

$$\begin{aligned} &A_1 = a_1 = b_1 = a_2 = 0, \quad k(y,t) = C_1, \\ &B_1 = \pm 2\sqrt{q_0}C_1, \quad A_0 = -\frac{1}{C_1}\frac{\mathrm{d}f_3}{\mathrm{d}t}, \\ &a_0 = f_1(y), \quad b_2 = 2q_0(1+f_1(y))/q_2, \\ &\eta(y,t) = -\int \frac{(1+f_1(y))\mathrm{d}y}{q_2 C_1} + f_3(t). \end{aligned} \tag{7}$$

Case 3:

$$\begin{aligned} &a_1 = b_1 = 0, \quad A_0 = -\frac{1}{C_1}\frac{\mathrm{d}f_2(t)}{\mathrm{d}t}, \\ &A_1 = \pm 2C_1\sqrt{q_4}, \quad B_1 = -A_1\sqrt{\frac{q_0}{q_4}}, \\ &a_0 = f_1(y), \quad a_2 = \frac{2(1+f_1(y))q_4}{q_2 + 2\sqrt{q_0}\sqrt{q_4}}, \\ &b_2 = \frac{q_0}{q_4}a_2, \quad k(y,t) = C_1, \\ &\eta(y,t) = -\int \frac{(1+f_1(y))\mathrm{d}y}{(q_2 + 2\sqrt{q_0}\sqrt{q_4})C_1} + f_2(t). \end{aligned} \tag{8}$$

Case 4:

$$\begin{aligned} &a_1 = b_1 = 0, \quad A_0 = -\frac{1}{C_2}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t}, \\ &A_1 = \pm 2C_2\sqrt{q_4}, \quad B_1 = A_1\sqrt{\frac{q_0}{q_4}}, \\ &a_0 = f_1(y), \quad a_2 = \frac{2(1+f_1(y))q_4}{q_2 - 2\sqrt{q_0}\sqrt{q_4}}, \\ &b_2 = \frac{q_0}{q_4}a_2, \quad k(y,t) = C_2, \\ &\eta(y,t) = -\int \frac{(1+f_1(y))\mathrm{d}y}{(q_2 - 2\sqrt{q_0}\sqrt{q_4})C_2} + f_3(t). \end{aligned} \tag{9}$$

Case 5:

$$\begin{aligned} &a_1 = b_1 = 0, \quad A_0 = -\frac{1}{C_2}\frac{\mathrm{d}f_1(t)}{\mathrm{d}t}, \\ &A_1 = \pm 2C_2\sqrt{q_4}, \quad B_1 = -A_1\sqrt{\frac{q_0}{q_4}}, \\ &a_0 = C_1, \quad a_2 = \frac{2(1+C_1)q_4}{q_2 + 2\sqrt{q_0}\sqrt{q_4}}, \\ &b_2 = \frac{q_0}{q_4}a_2, \quad k(y,t) = C_2, \\ &\eta(y,t) = \frac{-(1+C_1)y}{(q_2 + 2\sqrt{q_0}\sqrt{q_4})C_2} + f_1(t). \end{aligned} \tag{10}$$

Case 6:

$$\begin{aligned} &a_1 = b_1 = 0, \quad A_0 = -\frac{1}{C_3}\frac{\mathrm{d}f_2(t)}{\mathrm{d}t}, \\ &A_1 = \pm 2C_3\sqrt{q_4}, \quad B_1 = A_1\sqrt{\frac{q_0}{q_4}}, \\ &a_0 = C_1, \quad a_2 = \frac{2(1+C_1)q_4}{q_2 - 2\sqrt{q_0}\sqrt{q_4}}, \\ &b_2 = \frac{q_0}{q_4}a_2, \quad k(y,t) = C_3, \\ &\eta(y,t) = \frac{-(1+C_1)y}{(q_2 - 2\sqrt{q_0}\sqrt{q_4})C_3} + f_2(t). \end{aligned} \tag{11}$$

$f_1(y)$, $f_1(t)$, $f_2(t)$, $f_3(t)$ are arbitrary functions and C_1, C_2, C_3 are arbitrary constants.

Substituting Cases 1 – 6 into (5), we get the following formulas for the solutions of the (2+1)-dimensional DLW system:

$$u = -\frac{1}{C_1}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm 2\sqrt{q_4}C_1F(\xi),$$
$$v = f_1(y) + 2\frac{q_4(1+f_1(y))}{q_2}F^2(\xi),$$
$$\xi = xC_1 - \int \frac{(1+f_1(y))\mathrm{d}y}{q_2C_1} + f_3(t), \tag{12}$$

$$u = -\frac{1}{C_1}\frac{\mathrm{d}f_3}{\mathrm{d}t} \pm 2\sqrt{q_0}C_1F^{-1}(\xi),$$
$$v = f_1(y) + 2\frac{q_0(1+f_1(y))}{q_2}F^{-2}(\xi),$$
$$\xi = xC_1 - \int \frac{(1+f_1(y))\mathrm{d}y}{q_2C_1} + f_3(t), \tag{13}$$

$$u = -\frac{1}{C_1}\frac{\mathrm{d}f_2(t)}{\mathrm{d}t} \pm 2C_1[\sqrt{q_4}F(\xi) - \sqrt{q_0}F^{-1}(\xi)],$$
$$v = f_1(y) + \frac{2(1+f_1(y))}{q_2+2\sqrt{q_0}\sqrt{q_4}}[q_4F^2(\xi) + q_0F^{-2}(\xi)],$$
$$\xi = xC_1 - \int \frac{(1+f_1(y))\mathrm{d}y}{(q_2+2\sqrt{q_0}\sqrt{q_4})C_1} + f_2(t), \tag{14}$$

$$u = -\frac{1}{C_2}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm 2C_2[\sqrt{q_4}F(\xi) + \sqrt{q_0}F^{-1}(\xi)],$$
$$v = f_1(y) + \frac{2(1+f_1(y))}{q_2-2\sqrt{q_0}\sqrt{q_4}}[q_4F^2(\xi) + q_0F^{-2}(\xi)],$$
$$\xi = xC_2 - \int \frac{(1+f_1(y))\mathrm{d}y}{(q_2-2\sqrt{q_0}\sqrt{q_4})C_2} + f_3(t), \tag{15}$$

$$u = -\frac{1}{C_2}\frac{\mathrm{d}f_1(t)}{\mathrm{d}t} \pm 2C_2[\sqrt{q_4}F(\xi) - \sqrt{q_0}F^{-1}(\xi)],$$
$$v = C_1 + \frac{2(1+C_1)}{q_2+2\sqrt{q_0}\sqrt{q_4}}[q_4F^2(\xi) + q_0F^{-2}(\xi)],$$
$$\xi = xC_2 - \frac{(1+C_1)y}{(q_2+2\sqrt{q_0}\sqrt{q_4})C_2} + f_1(t), \tag{16}$$

$$u = -\frac{1}{C_3}\frac{\mathrm{d}f_2(t)}{\mathrm{d}t} \pm 2C_3[\sqrt{q_4}F(\xi) + \sqrt{q_0}F^{-1}(\xi)],$$
$$v = C_1 + \frac{2(1+C_1)}{q_2-2\sqrt{q_0}\sqrt{q_4}}[q_4F^2(\xi) + q_0F^{-2}(\xi)],$$
$$\xi = xC_3 - \frac{(1+C_1)y}{(q_2-2\sqrt{q_0}\sqrt{q_4})C_3} + f_2(t). \tag{17}$$

With the aid of the variable-coefficient extended F-expansion method and the formulas (12) – (17), we can obtain the exact solutions of (1) in terms of JEFs. Selecting the values of q_0, q_2, q_4 and the corresponding function F and inserting them into equations (12) and (13), we can construct the following periodic wave solutions of the (2+1)-dimensional DLW system:

$$u_1 = -\frac{1}{C_1}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm 2mC_1\mathrm{sn}\xi,$$
$$v_1 = f_1(y) - \frac{2m^2(1+f_1(y))}{(1+m^2)}\mathrm{sn}^2\xi,$$
$$u_2 = -\frac{1}{C_1}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm 2mC_1\mathrm{cd}\xi,$$
$$v_2 = f_1(y) - \frac{2m^2(1+f_1(y))}{(1+m^2)}\mathrm{cd}^2\xi,$$
$$\xi = xC_1 + \int \frac{[1+f_1(y)]\mathrm{d}y}{(1+m^2)C_1} + f_3(t), \tag{18}$$

$$u_3 = -\frac{1}{C_1}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm 2\mathrm{i}mC_1\mathrm{cn}\xi,$$
$$v_3 = f_1(y) - \frac{2m^2(1+f_1(y))}{(2m^2-1)}\mathrm{cn}^2\xi,$$
$$u_4 = -\frac{1}{C_1}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm 2\sqrt{1-m^2}C_1\mathrm{nc}\xi,$$
$$v_4 = f_1(y) + \frac{2(1-m^2)(1+f_1(y))}{(2m^2-1)}\mathrm{nc}^2\xi,$$
$$\xi = xC_1 - \int \frac{[1+f_1(y)]\mathrm{d}y}{(2m^2-1)C_1} + f_3(t), \tag{19}$$

$$u_5 = -\frac{1}{C_1}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm 2\mathrm{i}C_1\mathrm{dn}\xi,$$
$$v_5 = f_1(y) - \frac{2[1+f_1(y)]}{(2-m^2)}\mathrm{dn}^2\xi,$$
$$u_6 = -\frac{1}{C_1}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm 2\mathrm{i}\sqrt{1-m^2}C_1\mathrm{nd}\xi,$$
$$v_6 = f_1(y) - \frac{2(1-m^2)(1+f_1(y))}{(2-m^2)}\mathrm{nd}^2\xi,$$
$$\xi = xC_1 - \int \frac{[1+f_1(y)]\mathrm{d}y}{(2-m^2)C_1} + f_3(t), \tag{20}$$

$$u_7 = -\frac{1}{C_1}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm 2C_1\sqrt{1-m^2}\mathrm{sc}\xi,$$
$$v_7 = f_1(y) + \frac{2(1-m^2)[1+f_1(y)]}{(2-m^2)}\mathrm{sc}^2\xi,$$
$$u_8 = -\frac{1}{C_1}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm 2C_1\mathrm{cs}\xi,$$

(a) u_1 (b) v_1

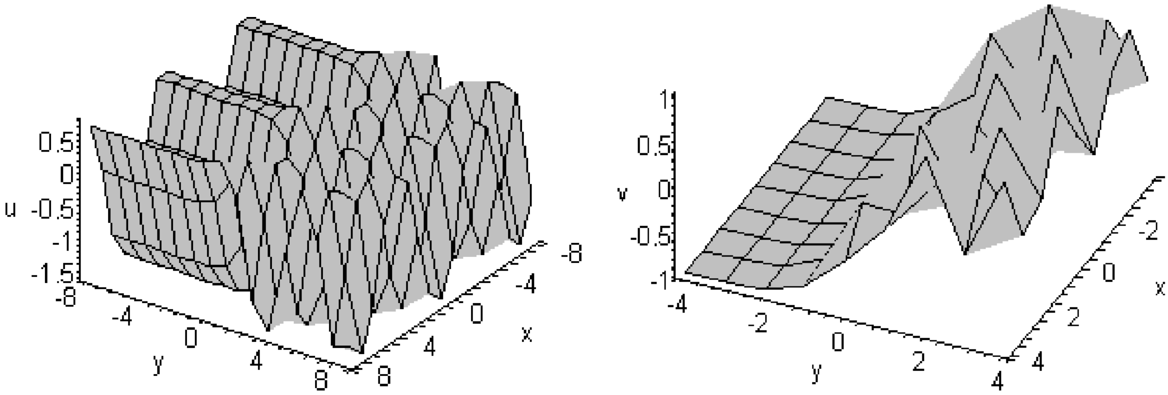

Fig. 1. Evolution plots of u_1 and v_1 in (18) with $m = 0.6$, $C_1 = 1$, $f_1(y) = \tanh(y)$, $f_3(t) = \tanh(t)$, $t = 1$.

(a) u_1 (b) v_1

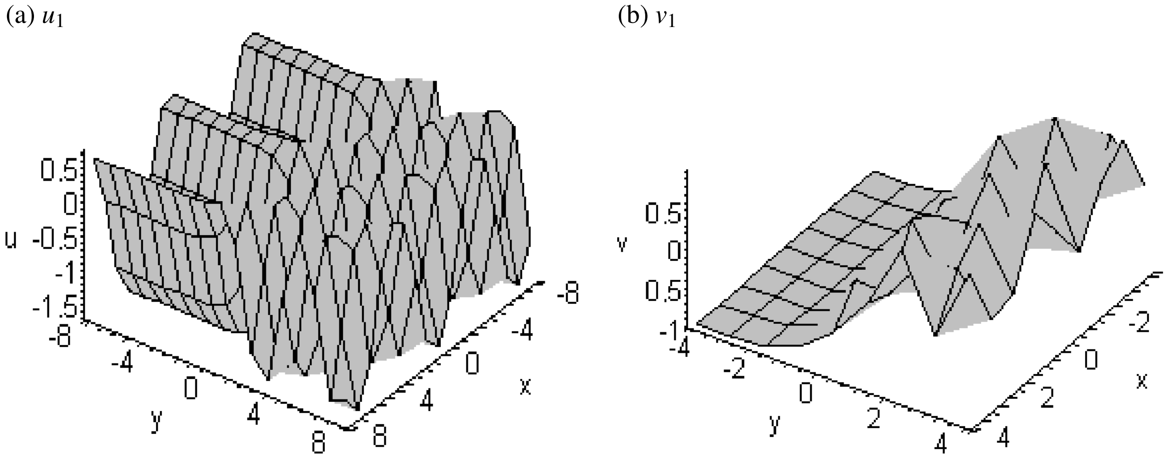

Fig. 2. Evolution plots of u_1 and v_1 with $m = 0.6$, $C_1 = 1$, $f_1(y) = \tanh(y)$, $f_3(t) = \mathrm{sn}(t, 0.5)$, $t = 1$.

(a) u_1 (b) v_1

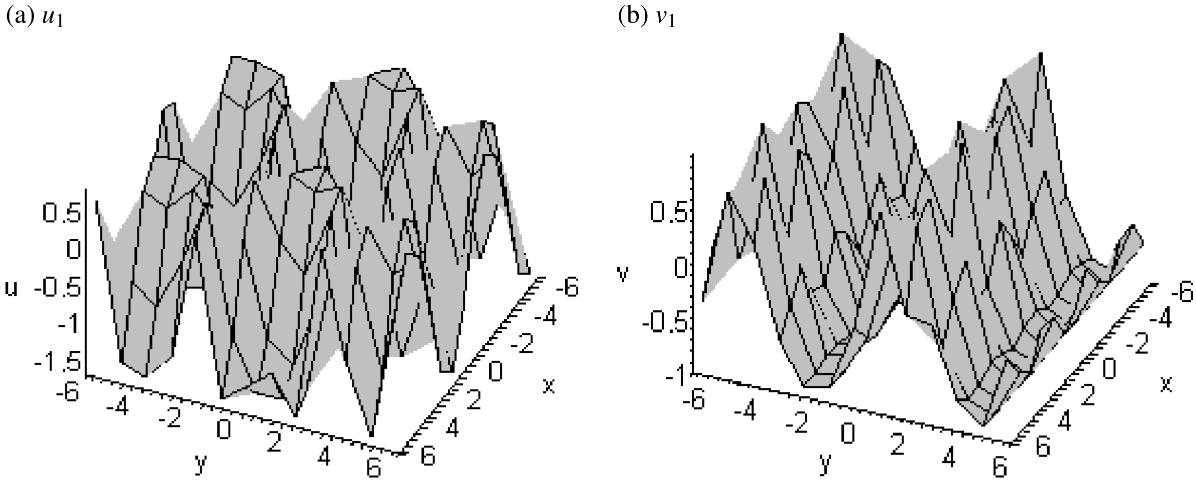

Fig. 3. Evolution plots of u_1 and v_1 with $m = 0.6$, $C_1 = 1$, $f_1(y) = \sin(y)$, $f_3(t) = \mathrm{sn}(t, 0.5)$ at $t = 1$.

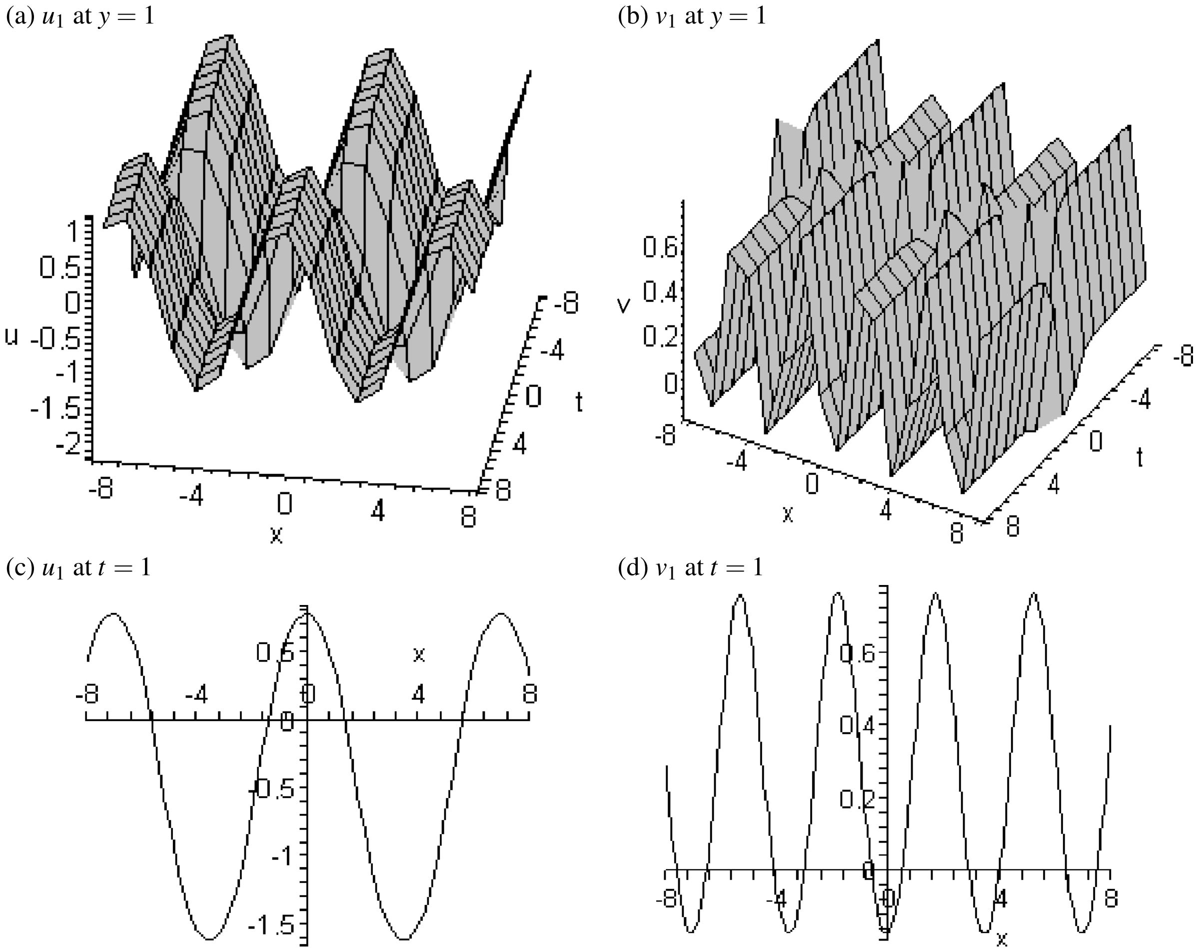

Fig. 4. The solution (u_1, v_1) of (18) at $y = 1$ with $m = 0.6$, $C_1 = 1$, $f_1(y) = \tanh(y)$, $f_3(t) = \tanh(t)$ and its position at $t = 1$.

$$v_8 = f_1(y) + \frac{2(1+f_1(y))}{(2-m^2)}\mathrm{cs}^2\xi,$$

$$\xi = xC_1 - \int \frac{[1+f_1(y)]\mathrm{d}y}{(2-m^2)C_1} + f_3(t). \tag{21}$$

We plotted the solution (u_1, v_1) of (18) to show the dynamics of the Jacobi elliptic wave solutions. The particular case of $f_1(y) = \tanh(y)$, $f_3(t) = \tanh(t)$, and "+" in each sign "±", see Figure 1. Moreover, we plotted (u_1, v_1) with various functions of $f_1(y)$ and $f_3(t)$ see Figures 2, 3. Figures 1 – 3 show the evolution plots of the solution (u_1, v_1). Then we can obtain various features of wave solutions depicted in Figures 1 – 3 by selecting arbitrary functions. In Figures 4 – 6 we plotted the periodic wave solution given by (u_1, v_1) at $y = 1$ and their positions at $t = 1$ with $m = 0.6$ and $C_1 = 1$.

Selecting $q_0 = 1/4$, $q_2 = (m^2-2)/2$, $q_4 = m^4/4$, we obtain

$$u_9 = -\frac{1}{C_1}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm m^2 C_1 \left(\frac{\mathrm{sn}\xi}{1 \pm \mathrm{dn}\xi}\right),$$

$$v_9 = f_1(y) - \frac{m^4[1+f_1(y)]}{2-m^2}\left[\frac{\mathrm{sn}\xi}{1 \pm \mathrm{dn}\xi}\right]^2,$$

$$u_{10} = -\frac{1}{C_1}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm C_1 \left(\frac{1 \pm \mathrm{dn}\xi}{\mathrm{sn}\xi}\right),$$

$$v_{10} = f_1(y) - \frac{(1+f_1(y))}{2-m^2}\left[\frac{1 \pm \mathrm{dn}\xi}{\mathrm{sn}\xi}\right]^2,$$

$$\xi = xC_1 + 2\int \frac{[1+f_1(y)]\mathrm{d}y}{(2-m^2)C_1} + f_3(t). \tag{22}$$

We plotted the solution (u_9, v_9) to show the evolution plots of solution. In Figure 7 we choose various functions $f_1(y)$ and $f_3(t)$ and take "+" in each

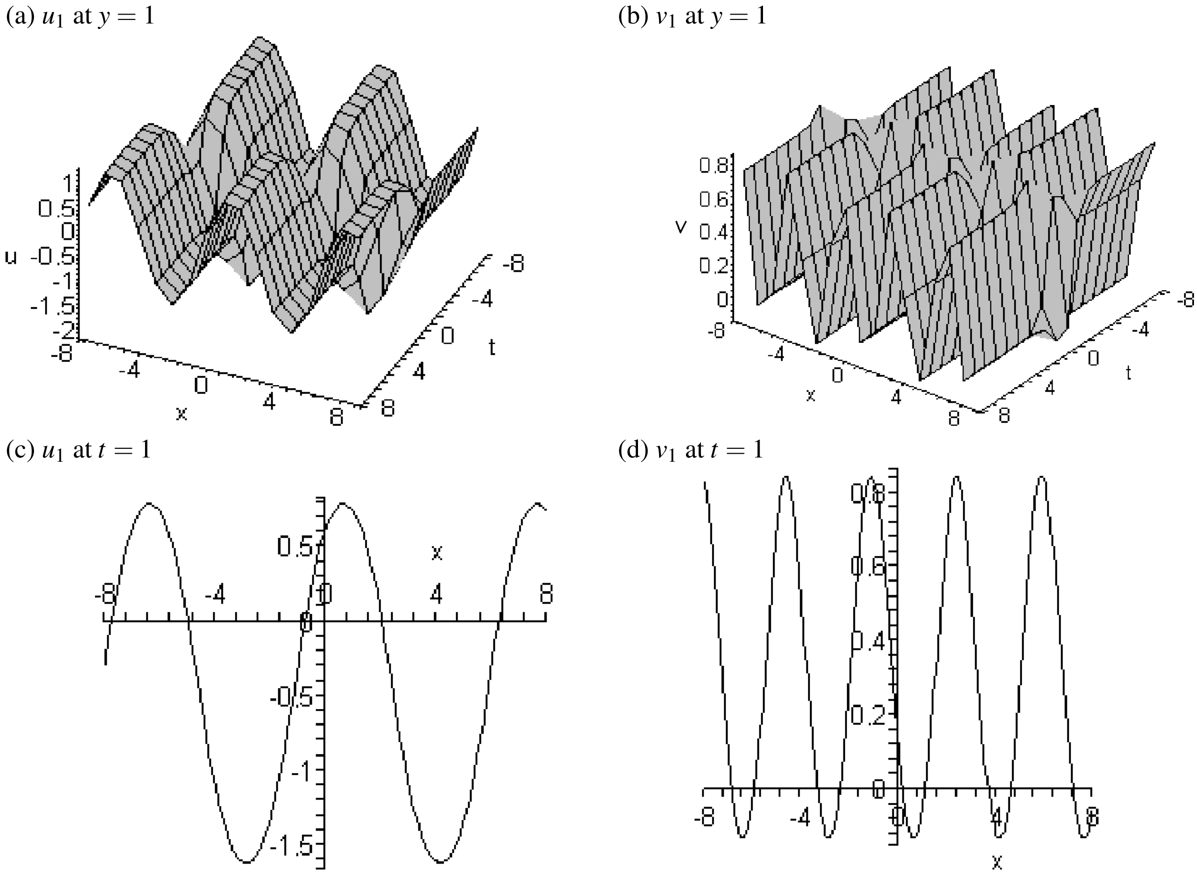

Fig. 5. The solution (u_1, v_1) of (18) at $y = 1$ with $m = 0.6$, $C_1 = 1$, $f_1(y) = \sin(y)$, $f_3(t) = \tanh(t)$ and its position at $t = 1$.

sign "±". We see that the solutions are affected by the selections of these functions. Then we can obtain many types of wave solutions as depicted in Figure 7. Also we plotted (u_9, v_9) at $y = 1$ and their positions at $t = 1$ with various functions $f_1(y)$ and $f_3(t)$ and $m = 0.6$, $C_1 = 1$ (see Figs. 8–9).

Selecting the values of q_0, q_2, q_4, and the corresponding function F and inserting them into equations (14)–(17), we can construct the following combined JEF solutions of (2+1)-dimensional DLW system:

$$
\begin{aligned}
u_{11} &= -\frac{1}{C_1}\frac{\mathrm{d}f_2(t)}{\mathrm{d}t} \pm 2C_1(m\,\mathrm{sn}\xi - \mathrm{ns}\xi),\\
v_{11} &= f_1(y) - \frac{2(1+f_1(y))}{(1-m)^2}\left[m^2\mathrm{sn}^2\xi + \mathrm{ns}^2\xi\right],\\
\xi &= xC_1 + \int \frac{[1+f_1(y)]\mathrm{d}y}{(1-m)^2 C_1} + f_2(t),
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
u_{12} &= -\frac{1}{C_2}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm 2C_2(m\,\mathrm{sn}\xi + \mathrm{ns}\xi),\\
v_{12} &= f_1(y) - \frac{2(1+f_1(y))}{(1+m)^2}\left[m^2\mathrm{sn}^2\xi + \mathrm{ns}^2\xi\right],\\
\xi &= xC_2 + \int \frac{[1+f_1(y)]\mathrm{d}y}{(1+m)^2 C_2} + f_3(t),
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
u_{13} &= -\frac{1}{C_1}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm 2\mathrm{i}C_1(\mathrm{dn}\xi - \sqrt{1-m^2}\,\mathrm{nd}\xi),\\
v_{13} &= f_1(y) - \frac{2(1+f_1(y))}{(2-m^2-2\sqrt{1-m^2})}\cdot[\mathrm{dn}^2\xi + (1-m^2)\mathrm{nd}^2\xi],\\
\xi &= xC_1 - \int \frac{[1+f_1(y)]\mathrm{d}y}{(2-m^2-2\sqrt{1-m^2})C_1} + f_3(t),
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
u_{14} &= -\frac{1}{C_2}\frac{\mathrm{d}f_2(t)}{\mathrm{d}t} \pm 2\mathrm{i}C_2(\mathrm{dn}\xi + \sqrt{1-m^2}\,\mathrm{nd}\xi),\\
v_{14} &= f_1(y) - \frac{2(1+f_1(y))}{(2-m^2+2\sqrt{1-m^2})}\cdot[\mathrm{dn}^2\xi + (1-m^2)\mathrm{nd}^2\xi],
\end{aligned}
$$

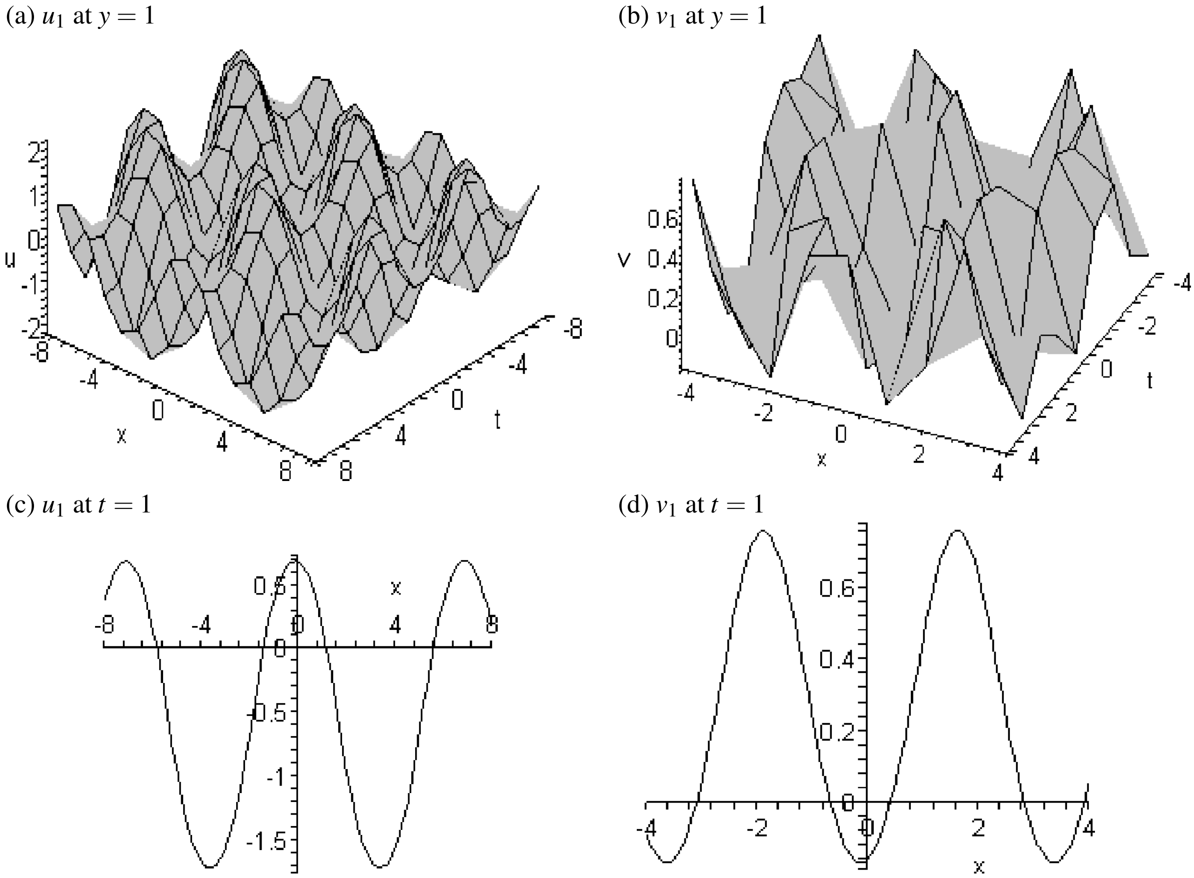

Fig. 6. The solution (u_1, v_1) of (18) at $y=1$ with $m=0.6$, $C_1=1$, $f_1(y)=\tanh(y)$, $f_3(t)=\mathrm{sn}(t,0.5)$ and its position at $t=1$.

$$\xi = xC_2 - \int \frac{[1+f_1(y)]\mathrm{d}y}{(2-m^2+2\sqrt{1-m^2})C_2} + f_2(t), \qquad (26)$$

$$u_{15} = -\frac{1}{C_1}\frac{\mathrm{d}f_2(t)}{\mathrm{d}t} \pm C_1\left(\frac{m^2\mathrm{sn}\xi}{1\pm\mathrm{dn}\xi} - \frac{1\pm\mathrm{dn}\xi}{\mathrm{sn}\xi}\right),$$

$$v_{15} = f_1(y) - \frac{[1+f_1(y)]}{2(1-m^2)} \cdot \left[(\frac{m^2\mathrm{sn}\xi}{1\pm\mathrm{dn}\xi})^2 + (\frac{1\pm\mathrm{dn}\xi}{\mathrm{sn}\xi})^2\right],$$

$$\xi = xC_1 + \int \frac{[1+f_1(y)]\mathrm{d}y}{(1-m^2)C_1} + f_2(t), \qquad (27)$$

$$u_{16} = -\frac{1}{C_2}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm C_2\left(\frac{m^2\mathrm{sn}\xi}{1\pm\mathrm{dn}\xi} + \frac{1\pm\mathrm{dn}\xi}{\mathrm{sn}\xi}\right),$$

$$v_{16} = f_1(y) - \frac{[1+f_1(y)]}{2} \cdot \left[(\frac{m^2\mathrm{sn}\xi}{1\pm\mathrm{dn}\xi})^2 + (\frac{1\pm\mathrm{dn}\xi}{\mathrm{sn}\xi})^2\right],$$

$$\xi = xC_2 + \int \frac{[1+f_1(y)]\mathrm{d}y}{C_2} + f_3(t). \qquad (28)$$

Other JEF solutions are omitted here for simplicity. When $m \to 1$, we have the hyperbolic function solutions as follows:

$$u_{17} = -\frac{1}{C_1}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm 2C_1 \tanh\xi,$$

$$v_{17} = f_1(y) - (1+f_1(y))\tanh^2\xi,$$

$$\xi = xC_1 + \int \frac{[1+f_1(y)]\mathrm{d}y}{2C_1} + f_3(t), \qquad (29)$$

$$u_{18} = -\frac{1}{C_1}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm 2\mathrm{i}C_1\mathrm{sech}\xi,$$

$$v_{18} = f_1(y) - 2(1+f_1(y))\mathrm{sech}^2\xi,$$

$$\xi = xC_1 - \int \frac{(1+f_1(y))dy}{C_1} + f_3(t), \qquad (30)$$

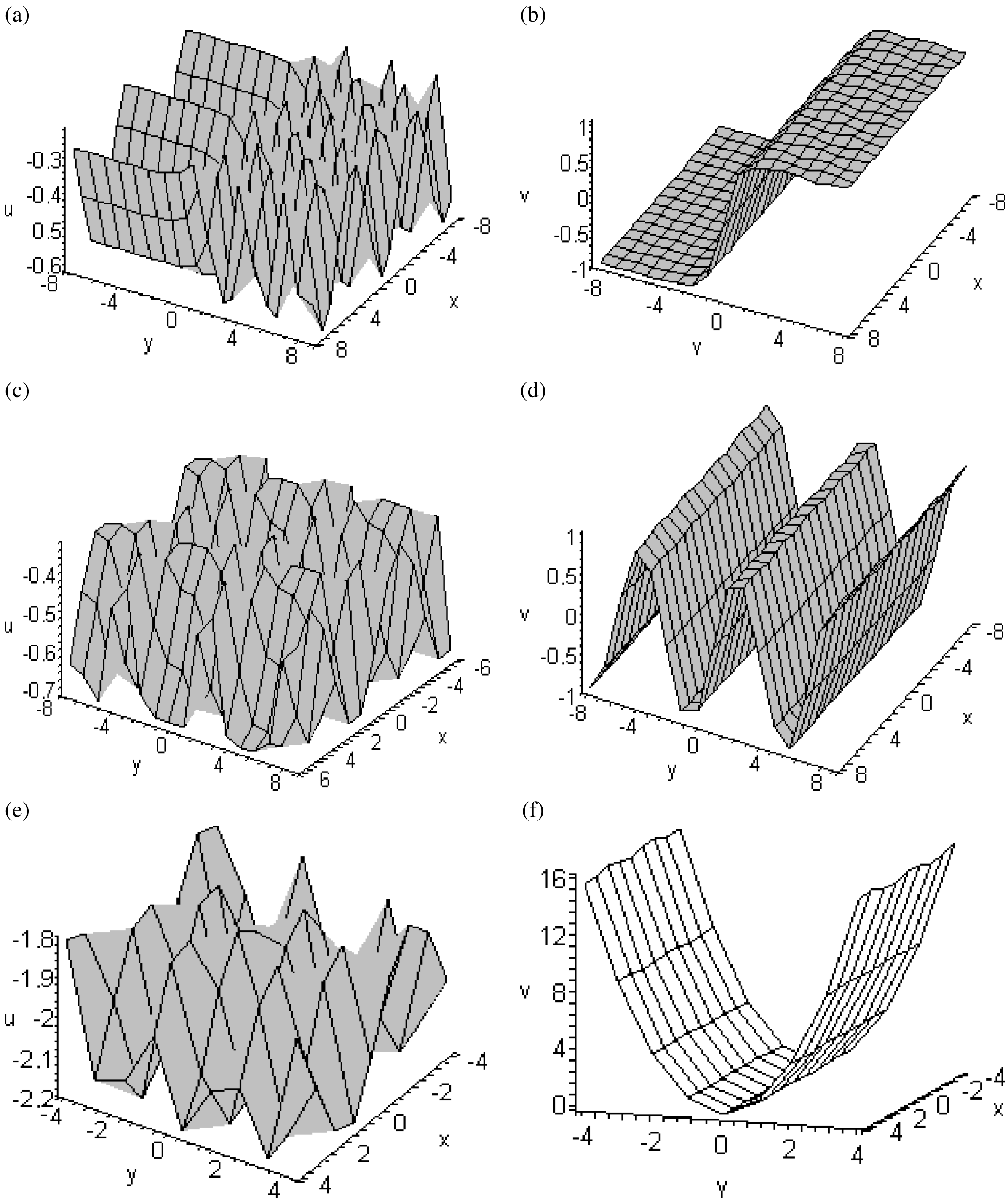

Fig. 7. The evolution plots of u_9 and v_9 in (22) with $m = 0.6$, $C_1 = 1$, $t = 1$. (a), (b): $f_1(y) = \tanh(y)$, $f_3(t) = \tanh(t)$. (c), (d): $f_1(y) = \sin(y)$, $f_3(t) = \mathrm{sn}(t, 0.5)$. (e), (f): $f_1(y) = y^2$, $f_3(t) = t^2$.

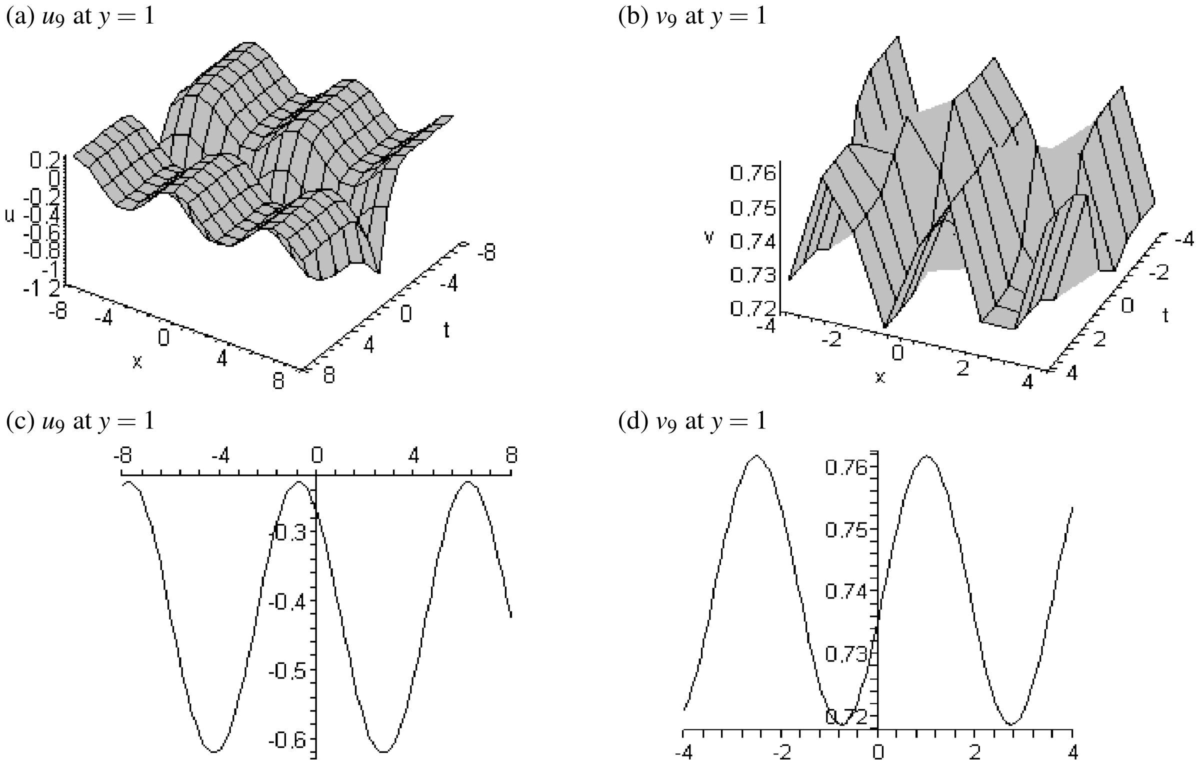

Fig. 8. The solution (u_9, v_9) at $y = 1$ with $m = 0.6$, $C_1 = 1$, $f_1(y) = \tanh(y)$, $f_3(t) = \tanh(t)$ and its position at $t = 1$.

$$
\begin{aligned}
u_{19} &= -\frac{1}{C_2}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm 2C_2[\tanh\xi + \coth\xi],\\
v_{19} &= f_1(y) - \frac{1}{2}(1+f_1(y))[\tanh^2\xi + \coth^2\xi],\\
\xi &= xC_2 + \int \frac{[1+f_1(y)]\mathrm{d}y}{4C_2} + f_3(t),
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
u_{20} &= -\frac{1}{C_1}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm C_1\left(\frac{\tanh\xi}{1+\mathrm{sech}\xi}\right),\\
v_{20} &= f_1(y) - (1+f_1(y))\left(\frac{\tanh\xi}{1+\mathrm{sech}\xi}\right)^2,\\
\xi &= xC_1 + 2\int \frac{[1+f_1(y)]\mathrm{d}y}{C_1} + f_3(t).
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
u_{21} &= -\frac{1}{C_2}\frac{\mathrm{d}f_3(t)}{\mathrm{d}t} \pm C_2\left(\frac{\tanh\xi}{1\pm\mathrm{sech}\xi} + \frac{1\pm\mathrm{sech}\xi}{\tanh\xi}\right),\\
v_{21} &= f_1(y) - \frac{[1+f_1(y)]}{2}\\
&\quad \cdot\left[\left(\frac{\tanh\xi}{1\pm\mathrm{sech}\xi}\right)^2 + \left(\frac{1\pm\mathrm{sech}\xi}{\tanh\xi}\right)^2\right],\\
\xi &= xC_2 + \int \frac{[1+f_1(y)]\mathrm{d}y}{C_2} + f_3(t).
\end{aligned} \tag{33}
$$

We used the solution (u_{15}, v_{15}) in (27) to show the evolution plots (see Figs. 10–11). When $m \to 0$, we can also obtain trigonometric function solutions of the system (1), we omit them here for simplicity. The solutions (30) and (31) are coincide with the results obtained in [27] and solutions (19) and (29) are found in [28]. Compared with the solutions obtained in [27, 28], here we further get many other exact solutions of (1). To show the properties of the solutions of the (2+1)-dimensional DLW system, we take the solutions (18), (22) and (27) as illustrative examples and show their plots (see Figs. 1–11).

4. Conclusion

The variable-coefficient extended F-expansion method have been used to construct more types of exact solutions of the (2+1)-dimensional DLW system. We have obtained many new solutions including single and combined JEFs, rational solutions, hyperbolic function solutions, and trigonometric function

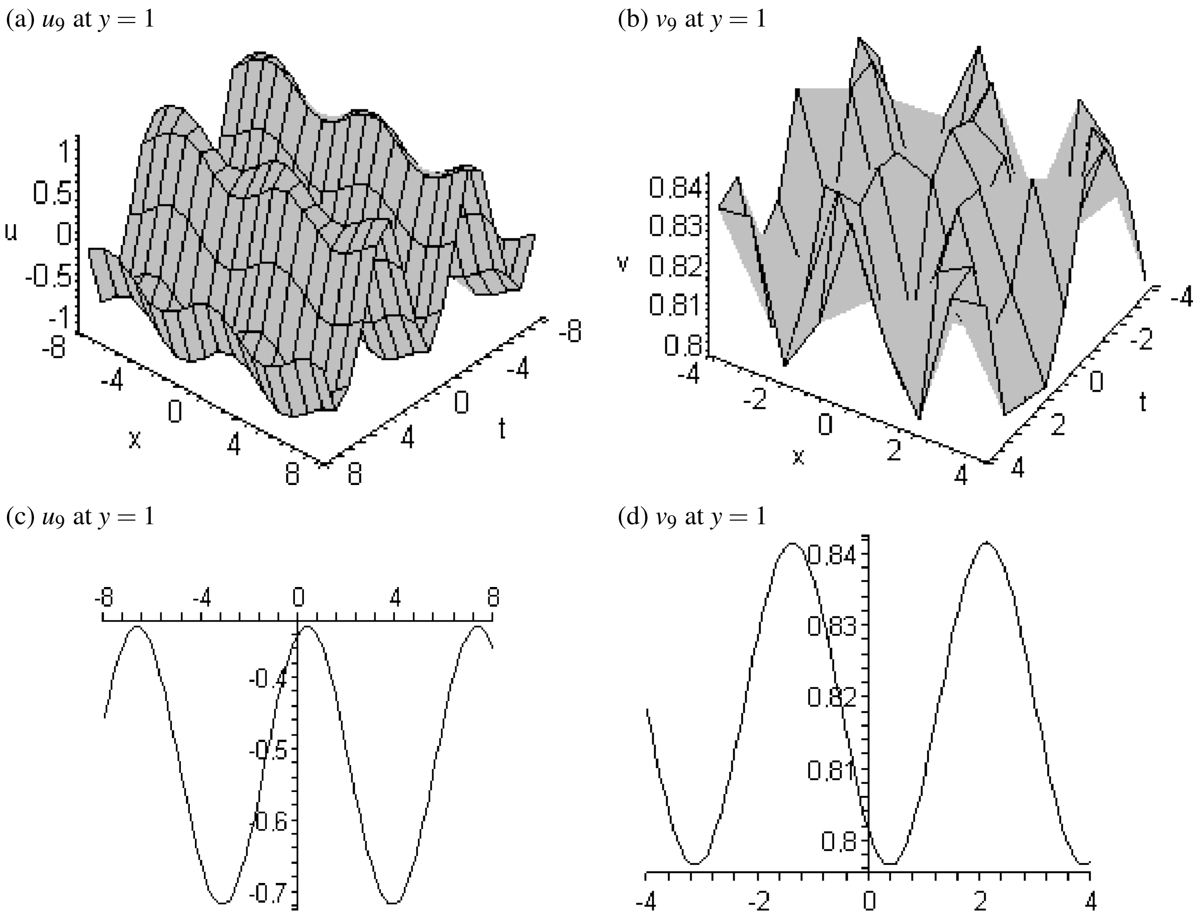

Fig. 9. The solution (u_9, v_9) at $y = 1$ with $m = 0.6$, $C_1 = 1$, $f_1(y) = \sin(y)$, $f_3(t) = \mathrm{sn}(t, 0.5)$ and its position at $t = 1$.

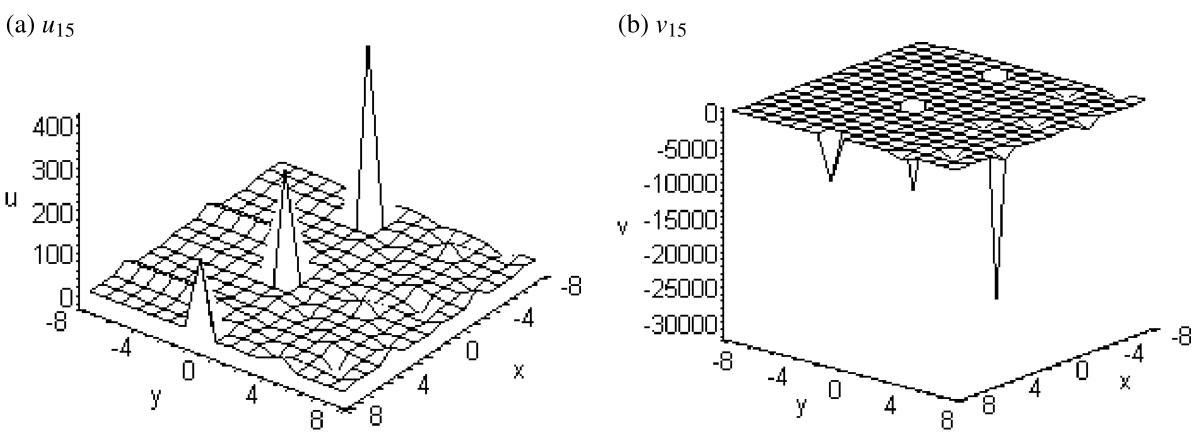

Fig. 10. Evolution plots of u_{15} and v_{15} in (27) with $m = 0.6$, $C_1 = 1$, $f_1(y) = \tanh(y)$, $f_2(t) = \tanh(t)$, $t = 1$.

(a) u_{15} (b) v_{15}

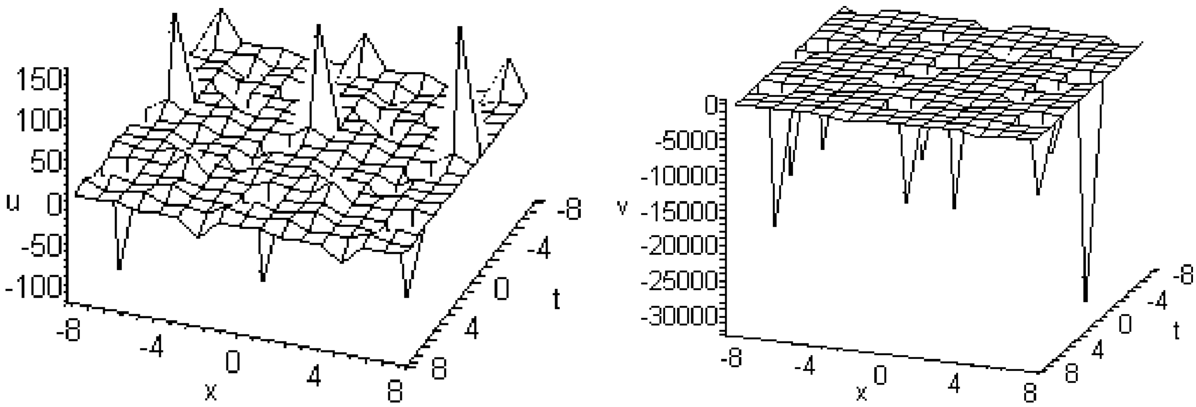

Fig. 11. Evolution plots of u_{15} and v_{15} in (27) with $m = 0.6$, $C_1 = 1$, $f_1(y) = \tanh(y)$, $f_2(t) = \mathrm{sn}(t, 0.5)$, $y = 1$.

solutions, each of which contains arbitrary functions and arbitrary constants. These solutions are more general than the previous solutions [28, 29] derived by other methods. The arbitrary functions in the obtained solutions imply that these solutions have rich local structures. The obtained solutions, which are accurate and explicit, may be important to explain some physical phenomena. We also hope that these solutions will be useful in further numerical analysis of solutions of the (2+1)-dimensional DLW system and will be helpful for a better understanding of the processes in such a system. The result given in this work shows that symbolic computations is a powerful tool in the process of seeking solutions of NLPDEs. Limit cases are studied and soliton-like solutions are got. The properties of the solutions of system (1) have been shown by means of their figures. It is shown that this method provides a very effective and powerful mathematical tool for solving many NLEEs in mathematical physics.

[1] M. J. Ablowitz and P. A. Clarkson, Soliton, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, New York 1991.
[2] W. Malfiet, Am. J. Phys. **60**, 650 (1992); W. Malfiet and W. Hereman, Phys. Scr. **54**, 563 (1996).
[3] W. Malfliet, Math. Computers in Simulation **62**, 101 (2003).
[4] A. H. Khater, W. Malfiet, D. K. Callebaut, and E. S. Kamel, Chaos, Solitons, and Fractals **14**, 513 (2002); A. H. Khater, D. K. Callebaut, W. Malfiet, and E. S. Kamel, J. Comput. Appl. Math. **140**, 435 (2002).
[5] Y. T. Gao and B. Tian, Comput. Phys. Commun. **133**, 158 (2001).
[6] E. Fan, Phys. Lett. A **277**, 212 (2000).
[7] J. H. He and X. H. Wu, Chaos, Solitons, and Fractals **30**, 700 (2006).
[8] A. M. Wazwaz, Comput. Math. Appl. **47**, 583 (2004); E. Yusufoglu, A. Bekir, and M. Alp, Chaos, Solitons, and Fractals **37**, 1193 (2008).
[9] S. K. Liu, Z. T. Fu, S. D. Liu, and Q. Zhao, Phys. Lett. A **289**, 69 (2001); Z. T. Fu, S. K. Liu, S. D. Liu, and Q. Zhao, Phys. Lett. A **290**, 72 (2001).
[10] E. J. Parkes, B. R. Duffy, and P. C. Abbott, Phys. Lett. A **295**, 280 (2002).
[11] H. T. Chen and H. Q. Zhang, Chaos, Solitons, and Fractals **15**, 585 (2003).
[12] Y. Z. Peng, Chin. J. Phys. **41**, 103 (2003); Y. Z. Peng, Phys. Lett. A **314**, 401 (2003).
[13] E. Yomba, Chaos, Solitons, and Fractals **21**, 209 (2004).
[14] Y. B. Zhou, M. L. Wang, and Y. M. Wang, Phys. Lett. A **308**, 31 (2003).
[15] J. Liu and K. Yang, Chaos, Solitons, and Fractals **22**, 111 (2004); E. Yomba, Phys. Lett. A **340**, 149 (2005); M. Wang and X. Li, Phys. Lett. A **343**, 48 (2005).
[16] Y. J. Ren and H. Q. Zhang, Chaos, Solitons, and Fractals **27**, 959 (2006).
[17] S. Zhang and T. Xia, Commun. Nonlinear Sci. Numer. Simulation **13**, 1294 (2008).
[18] H. T. Chen and H. Q. Zhang, Chaos, Solitons, and Fractals **20**, 765 (2004); H. T. Chen and H. Q. Zhang, Chinese Phys. **12**, 1202 (2003).
[19] M. M. Hassan and A. H. Khater, Physica A **387**, 2433 (2008).

[20] J.-F. Zhang, C.-Q. Dai, Q. Yang, and J.-M. Zhu, Optics Commun. **252**, 408 (2005).
[21] E. G. Fan and J. Zhang, Phys. Lett. A **305**, 383 (2002).
[22] Z. Y. Yan, Z. Naturforsch. **59a**, 23 (2004).
[23] A. H. Khater and M. M. Hassan, Z. Naturforsch. **59a**, 389 (2004); A. H. Khater, M. M. Hassan, and R. S. Temsah, J. Phys. Soc. Japan **74**, 1431 (2005).
[24] M. Boiti, J. J. P. Leon, and F. Pempinelli, Inverse Problem **3**, 371 (1987).
[25] G. Paquin and P. Winternitz, Physica D **46**, 122 (1990).
[26] E. Yomba, Chaos, Solitons, and Fractals **20**, 1135 (2004); E. G. Fan, Chaos, Solitons, and Fractals **16**, 819 (2003).
[27] Y. Chen and B. Li, Int. J. Eng. Sci. **42**, 715 (2004).
[28] Y. Chen and Q. Wang, Chaos, Solitons, and Fractals **23**, 801 (2005).
[29] S. Zhang, Chaos, Solitons, and Fractals **32**, 847 (2007).
[30] C. L. Bai and H. Zhao, Chin. J. Phys. **44**, 94 (2006).
[31] Q. Wang, Y. Chen, and H. Q. Zhang, Appl. Math. Comput. **181**, 247 (2006).
[32] Y. Chen and X. Yong, Int. J. Nonlinear Sci. **4**, 147 (2007).
[33] W.-L. Zhang, G.-J. Wu., M. Zhang, J.-M. Wang, and J.-H. Han, Chinese Phys. B **17**, 1156 (2008).
[34] S. Zhang, W. Wang, and J.-L. Tong, Electronic J. Theor. Phys. **5**, 117 (2008).
[35] C. L. Bai and H. J. Niu, Eur. Phys. J. D **47**, 221 (2008).
[36] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover, New York 1965.