Novel Excitations of the Nonlinear Schrödinger Equation by Separation of Variables

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By means of an extended tanh method, a new type of variable separation solutions with two arbitrary lower-dimensional functions of the (2+1)-dimensional nonlinear Schrödinger (NLS) equation is derived. Based on the derived variable separation excitation, some special types of localized solutions such as a curved soliton, a straight-line soliton and a periodic soliton are constructed by choosing appropriate functions. In addition, one dromion changes its shape during the collision with a folded solitary wave.

Key words: Variable Separation Solution; Extended Tanh Method; (2+1)-Dimensional Nonlinear Schrödinger Equation.
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1. Introduction

Solitons appear in almost all branches of physics, such as hydrodynamics, plasma physics, nonlinear optics, condensed matter physics, low temperature physics, particle physics, nuclear physics, biophysics and astrophysics. The study of solitons is a systematic research on nonlinear phenomena with a consistent leading principle. While the soliton concept gives a new point of view on nature, there are many problems to be studied. The soliton concept has been developed with many approaches such as the Bäcklund transformation [1], the Darboux transformation [2], the Cole-Hopf transformation [3], various tanh methods [4], various Jacobi elliptic function methods [5, 6], multilinear variable separation approaches (MLVSA) [7, 8], the Painlevé method [9], the homogeneous balance method [10], and the similarity reduction method [11]. Among them, the extended tanh method (ETM) is a useful approach to obtain variable separation solutions for (2+1)-dimensional systems.

For a given nonlinear evolution equation

$$\Lambda(U, U_t, U_{x_i}, U_{x_i x_j}, \ldots) = 0 \tag{1}$$

with independent variables $\varsigma = (t, x_1, x_2, \ldots, x_m)$ and the dependent variable U, we seek its solutions in the form

$$U = \sum_{i=0}^{n} a_i(\varsigma)\phi^i(\omega(\varsigma)), \quad \omega(\varsigma) = \sum_{i=0}^{m} g_i x_i. \tag{2}$$

Using relation (2), one obtains many explicit and exact travelling wave solutions of nonlinear evolution equations. The main idea of the approach is that $\phi(\omega(\varsigma))$ is assumed to be a solution of some equations such as the quartic nonlinear Klein-Gordon equation $\phi'^2 = \alpha_4\phi^4 + \alpha_2\phi^2 + \alpha_0$, or a solution of the general elliptic equation $\phi'^2 = \sum_{i=0}^{4} \alpha_i\phi^i$, where $\alpha_i, i \in (1,2,3,4)$ are all arbitrary constants.

In the ETM [12 – 15], $\phi(\omega(\varsigma))$ is assumed to be a solution of the equation

$$\phi' = \alpha_0 + \phi^2. \tag{3}$$

Different from (2) is that $\omega(\varsigma)$ is not a simple linear combination of the variables x_i, but is assumed to be an arbitrary function with the variable separated form

$$\omega(\varsigma) = \varsigma_1(x_1, t) + \varsigma_2(x_2, t) + \varsigma_3(x_3, t) + \ldots, \tag{4}$$

where the ς_i are arbitrary functions of the indicated variables.

To determine U explicitly, one may take the following steps: First, similar to the usual tanh approach, one

determines n by balancing the highest-order nonlinear terms and the highest-order partial derivative terms in the given nonlinear evolution equation. Second, substituting (2), (3) and (4) into the given equation and collecting coefficients of polynomials of ϕ, then eliminating each coefficient allows to derive a set of partial differential equations for the a_i $(i = 0,1,\ldots,n)$ and ω. Third, solving the system of partial differential equations one obtains the a_i and ω. Finally, as (3) with $\alpha_0 = 0$ possesses the solution

$$\phi = -\frac{1}{\omega}, \tag{5}$$

substituting a_i, ω and (5) into (2), one can obtain the solution of the equation in concern.

In the present paper, with the help of the ETM we get variable separated solutions for the (2+1)-dimensional nonlinear Schrödinger (NLS) equation. Some special types of soliton solutions and periodic soliton solutions are constructed by choosing appropriate functions in the general variable separation solution of this system. In addition, through fixing the parameters further, we manage to obtain a new type of evolutionary and interaction properties for dromion and folded solitary waves. These may change their shapes during the collision.

2. ETM and Variable Separation Solution for the (2+1)-Dimensional NLS Equation

The integrable (2+1)-dimensional NLS equation [16] is

$$U_{xx} = UQ_x - \mathrm{i}U_t,\ U^*_{xx} = U^*Q_x + \mathrm{i}U^*_t,\ Q_y = UU^*. \tag{6}$$

Here, U is a complex function of the real variables x, y and t; $\mathrm{i}^2 = -1$. In the case $x = y$, this equation is reduced to

$$\mathrm{i}U_t + U_{xx} + 2|U|^2U = 0, \tag{7}$$

which is the celebrated NLS equation. The NLS equation has been widely used to study the dynamics of small but finite amplitude nonlinearly interacting perturbations in many-body physics, in nonlinear optics and optical communications, in nonlinear plasmas and complex geophysical flows, as well as in intense laser-plasma interactions and nonlinear quantum electrodynamics. For example, the NLS equation with a cubic nonlinearity is a suitable model for the nonlinear pulse propagation in Kerr media, photonics and optical fibre communications, as well as in unmagnetized plasmas. Equation (6) is the generalization of the nonisotropic Lax integrable (1+1)-dimensional NLS equation. And it may also be obtained from the symmetrical restriction of the Kadomtsev-Petviashvili (KP) equation [16]. Moreover, (6) can be transformed to Hirota-type equations:

$$\begin{aligned} &(D_x^2 + \mathrm{i}D_t)G\cdot F - Q_{0x}GF = 0,\\ &D_xD_yF\cdot F + GG^* = 0, \end{aligned} \tag{8}$$

with the transformation $U = G/F$, $Q = -2(\ln F)_x + Q_0(x,t)$. Here we have used Hirota's D-operators:

$$\begin{aligned} D_x^m\cdots D_y^nF\cdot G = (\partial_x - \partial'_x)^m\cdots(\partial_y - \partial'_y)^n\\ \cdot F(x,\cdots,y)G(x,\cdots,y)|_{x'=x,y'=y}. \end{aligned} \tag{9}$$

$Q_0 \equiv Q_0(x,t)$ is an arbitrary function of x,t. Using a special Bäcklund transformation and the MLVSA [17,18] one finds some special types of excitations such as dromions, breathers, instantons, ring-type solitons, fractal-dromions, fractal-lumps, peakons, compactons, and folded waves. It is worth noting that here we solve the NLS equation simply using the ETM instead of the MLVSA, and then get some novel excitations. In particular, we are interested in a concrete nonelastic interaction between a dromion and a folded solitary wave.

Along with the ETM, we assume that the system (6) possesses solutions of the form

$$\begin{aligned} U(x,y,t) &= \sum_{j=0}^{m} a_j\phi^j(\omega)\exp[\mathrm{i}(r+s)],\\ Q(x,y,t) &= \sum_{k=0}^{n} b_k\phi^k(\omega), \end{aligned} \tag{10}$$

where the real function ϕ satisfies

$$\phi' = \phi^2, \tag{11}$$

$r \equiv r(x,t)$ is a function of $\{x,t\}$, $s \equiv s(y,t)$ is a function of $\{y,t\}$, and $\omega \equiv \omega(x,y,t)$, $a_j \equiv a_j(x,y,t)$ $(j = 0,1,\ldots,l)$, $b_k \equiv b_k(x,y,t)$ $(k = 0,1,\ldots,m)$ are functions being determined later. By balancing the highest-order derivative terms with the nonlinear terms in system (6), we obtain $m = n = 1$. Then we have

$$\begin{aligned} U(x,y,t) &= a_0(x,y,t)\exp[\mathrm{i}(r+s)]\\ &\quad + a_1(x,y,t)\phi(\omega)\exp[\mathrm{i}(r+s)],\\ Q(x,y,t) &= b_0(x,y,t) + b_1(x,y,t)\phi(\omega). \end{aligned} \tag{12}$$

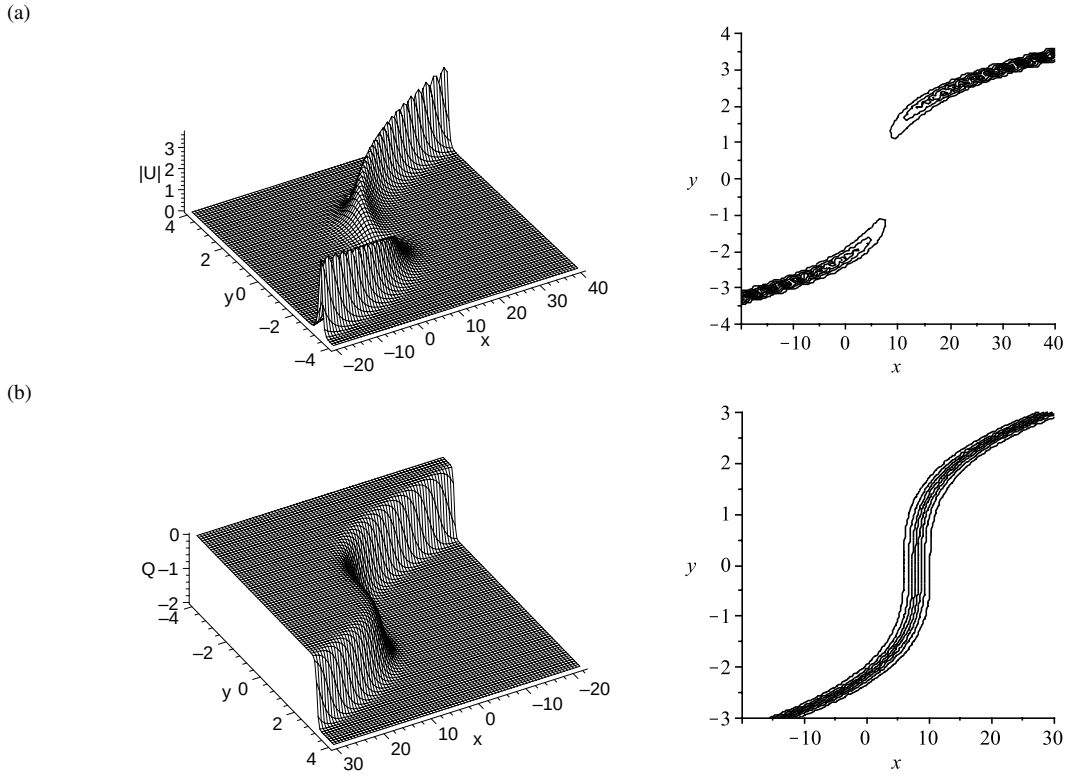

Fig. 1. The curve-type solitary wave solution for: (a) $|U|$ at $t=0$ determined by (25) with (31); (b) the field Q at $t=0$ determined by (26) with (31).

Inserting (11) and (12) into (6), selecting the variable separation ansatz

$$\omega = p(x,t) + q(y,t), \tag{13}$$

and eliminating all the coefficients of the polynomials in ϕ, one gets the following set of partial differential equations:

$$2a_1p_xr_x + a_1p_t + a_1q_t = 0, \tag{14}$$

$$2a_{1x}r_x + a_{1t} + a_1r_{xx} = 0, \tag{15}$$

$$a_{0t} + 2r_xa_{0x} + r_{xx}a_0 = 0, \tag{16}$$

$$2a_1p_x^2 - a_1b_1p_x = 0, \tag{17}$$

$$a_1p_{xx} + 2a_{1x}p_x - a_0b_1p_x - a_1b_{1x} = 0, \tag{18}$$

$$-a_1r_t - a_1r_x^2 + a_{1xx} - a_1b_{0x} - a_0b_{1x} - s_ta_1 = 0, \tag{19}$$

$$a_0b_{0x} - r_x^2a_0 + a_{0xx} - r_ta_0 - s_ta_0 = 0, \tag{20}$$

$$b_1q_y - a_1^2 = 0, \tag{21}$$

$$b_{1y} - 2a_1a_0 = 0, \tag{22}$$

$$b_{0y} - a_0^2 = 0. \tag{23}$$

Now, we are left to solve (14) to (23). By careful analysis and calculation, we obtain

$$\begin{aligned} &a_0 = 0, \quad a_1 = \sqrt{2p_xq_y}, \quad b_1 = 2p_x, \\ &b_0 = \int_0^x \left(\frac{2p_{xxx}p_x - p_{xx}^2 - 4p_x^2r_t - 4p_x^2r_x^2}{4p_x^2} \right) \mathrm{d}x + c_2, \\ &p = p(x,t), \quad q = Y(y) + T(t), \quad s = s(y), \\ &r = \int_0^x \left(-\frac{p_t + q_t}{2p_x} \right) \mathrm{d}x + c_1, \end{aligned} \tag{24}$$

where $p \equiv p(x,t)$ and $Y \equiv Y(y)$ are arbitrary functions of the indicated variables, and c_1 and c_2 are integration constants. Consequently, the exact variable separation solutions of the NLS equation (6) have the forms

$$U = \frac{\sqrt{2p_xq_y}}{p+q} \exp\left[\mathrm{i} \left(\int_0^x \left(-\frac{p_t + q_t}{2p_x} \right) \mathrm{d}x + s(y) + c_1 \right) \right], \tag{25}$$

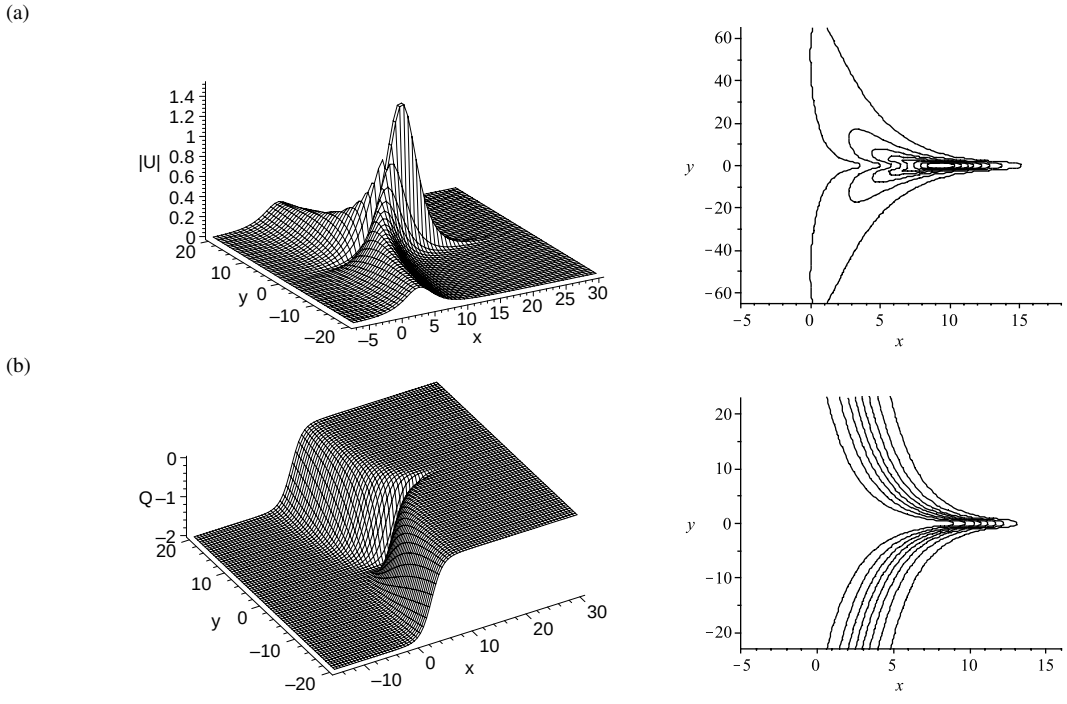

Fig. 2. The straight-line-type solitary wave solution with head for: (a) $|U|$ at $t=0$ determined by (25) with (32); (b) the field Q at $t=0$ determined by (26) with (32).

$$Q=\int_0^x\left(\frac{2p_{xxx}p_x-p_{xx}^2-4p_x^2r_t-4p_x^2r_x^2}{4p_x^2}\right)\mathrm{d}x + \frac{2p_x}{p+q}+c_2. \tag{26}$$

Obviously, one special choice for the allowed conditions is

$$\omega=p(x,t)+q(y)=X(x)+T(t)+Y(y), \tag{27}$$

where $X\equiv X(x)$, $T\equiv T(t)$, and $Y\equiv Y(y)$ are three arbitrary variable separation functions of x, t, and y, respectively. Under this special condition the time and space variables are separated completely. Then we have

$$U=\frac{\sqrt{2X_xY_y}}{X+Y+T}\exp\left[\mathrm{i}\left(\int_0^x\left(-\frac{T_t}{2X_x}\right)\mathrm{d}x+s+c_1\right)\right], \tag{28}$$

$$Q=\int_0^x\left(\frac{2X_{xxx}X_x-X_{xx}^2-4X_x^2r_t-4X_x^2r_x^2}{4X_x^2}\right)\mathrm{d}x + \frac{2X_x}{X+Y+T}+c_2. \tag{29}$$

It is worth emphasizing here that

$$UU^*=\frac{2p_xq_y}{(p+q)^2}, \tag{30}$$

which just shares the form of the so-called universal quantity. Therefore, similar to the ways in previous literature like [15], starting with the results (25) and (26), the (2+1)-dimensional NLS equation admits various localized excitations and interaction properties, such as curved-line solitons, ring solitons, dromions, peakons, compactons, foldons, chaotic solitons.

3. Special New Localized Excitations

In the following, we focus our attention on some new and interesting explicit solutions of the physical fields U and Q in the (2+1)-dimensional NLS equation, and list them as follows. To our knowledge, these new soliton structures of the NLS equation have not been reported previously in the literature [17, 18].

In Figs. 1 and 2, two special types of solitary waves are plotted for the fields U and Q determined by (25) and (26) with the chosen functions

$$p=\exp(x+\omega t-8)^{-1},\quad q=\exp(-0.8y^3),\quad c_2=0 \tag{31}$$

and

$$p=\cosh(x+\omega t-9)^{-1},\quad q=0.03+y^2,\quad c_2=0. \tag{32}$$

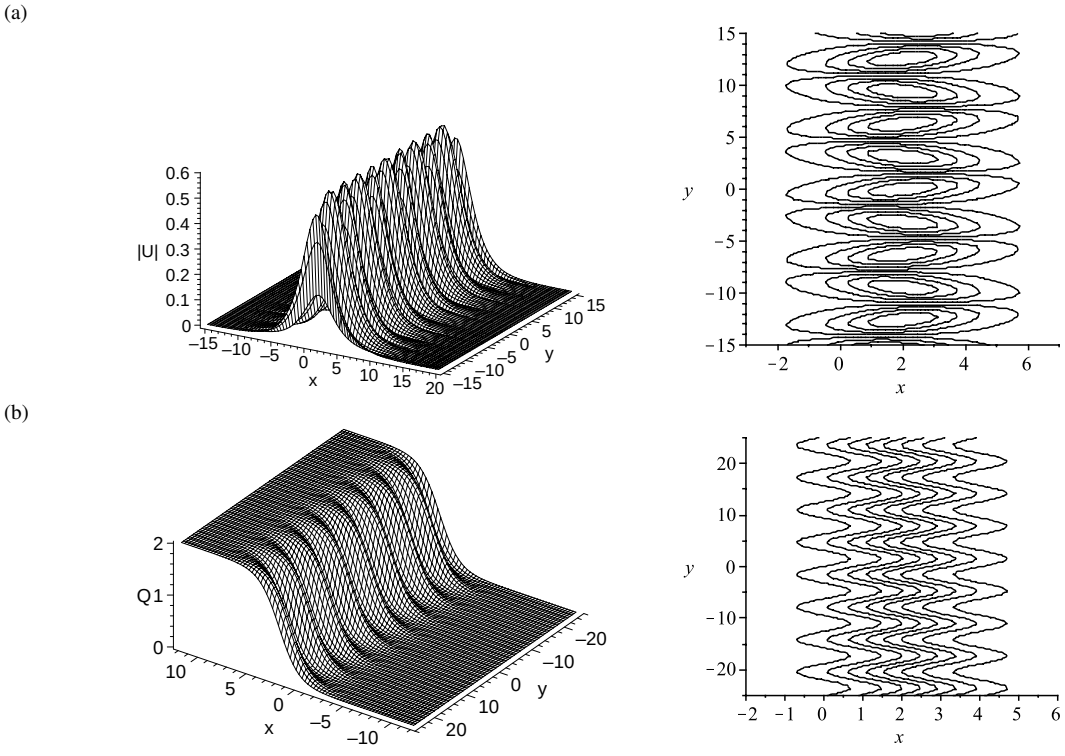

Fig. 3. Periodic solitary wave solution for: (a) $|U|$ at $t=0$ determined by (25) with (33); (b) the field Q at $t=0$ determined by (26) with (33).

As we know, a curved-line soliton of an integrable model is defined as a solution which is finite on a curved-line and decays exponentially away from the curve. A straight-line soliton of an integrable model is defined as a localized excitation possessing nonzero values for a suitable physical quantity, say energy, on a full straight-line and decaying exponentially away from the line. Thus straight-line solitons are special cases of curved ones.

Figure 3 shows another typical periodic solitary wave for the fields U and Q determined by (25) and (26) with the function choices

$$\begin{aligned} &p=\exp(-x+\omega t+2)^{-1},\\ &q=\exp[0.7\sin(y)], \quad c_2=0. \end{aligned} \tag{33}$$

4. Interaction between a Special Dromion and a Folded Solitary Wave

As a matter of fact, if p_x is taken as (1+1)-dimensional localized multi-value function, say a loop soliton,

$$\begin{aligned} &p_x\equiv\sum_{j=1}^{M}h_j(\xi-v_jt),\\ &x=\xi+\sum_{j=1}^{M}r_j(\kappa_jt)g_j(\xi-v_jt), \end{aligned} \tag{34}$$

and the function q_y is given in a similar way,

$$q_y=\sum_{j=1}^{M}Q_j(\eta), \quad y=\eta+\Omega(\eta), \tag{35}$$

where the v_j and κ_j are all arbitrary constants and the h_j, g_j are all localized functions with the properties

$$\begin{aligned} &h_j(\pm\infty)=H^{\pm}, \quad g_j(\pm\infty)=G^{\pm}=\text{ constant},\\ &r_j(\pm\infty)=R^{\pm}=\text{ constant}, \end{aligned} \tag{36}$$

then we have

$$p=\int^{\xi}p_xx_\xi\,\mathrm{d}\xi, \quad q=\int^{\eta}q_yy_\eta\,\mathrm{d}\eta. \tag{37}$$

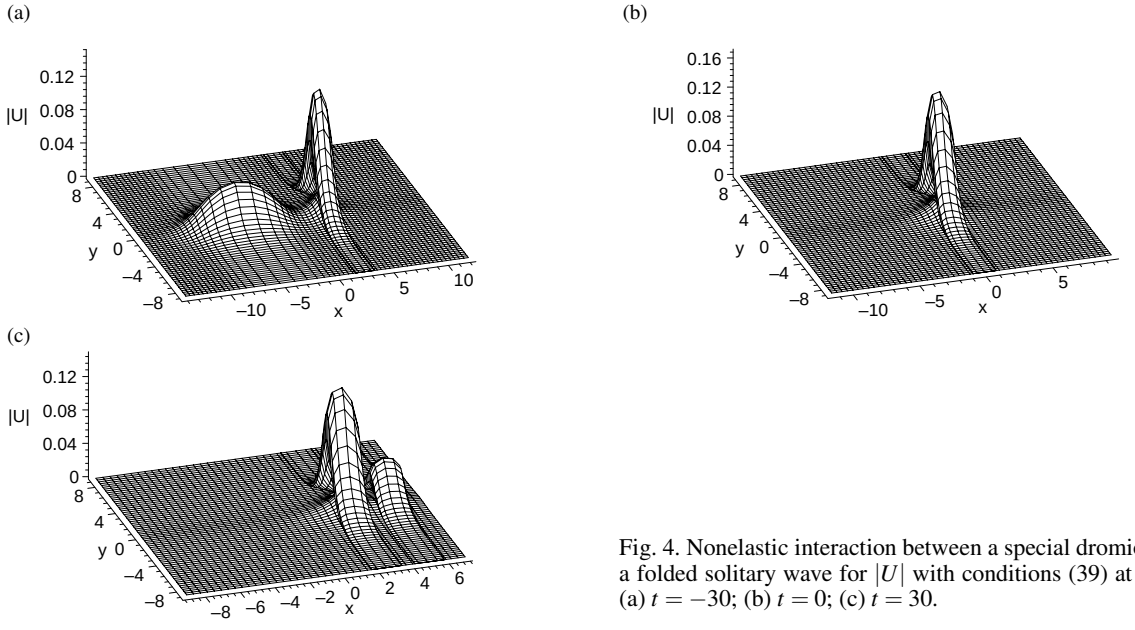

Fig. 4. Nonelastic interaction between a special dromion and a folded solitary wave for $|U|$ with conditions (39) at times: (a) $t=-30$; (b) $t=0$; (c) $t=30$.

Substituting (34)–(37) into (30), we can get some interesting coherent excitations for $|U|$. Judged from expression (34), ξ may be a multi-value function in certain regions of x by choosing the functions r_j and g_j suitably. Therefore, the function p_x may be a multi-valued function of x in these regions, though it is a single-valued function of ξ. Besides, p_x is an interacting travelling solution of M localized excitations due to the property $\xi|_{x\to\infty}\to x\to\infty$. In this case, the phase factors $\Delta_j^{\mp}$ read

$$\Delta_j(\pm)=\sum_{i<j}G_i^{\mp}R_i^{\mp}+\sum_{i>j}G_i^{\pm}R_i^{\pm}. \tag{38}$$

For instance, Figs. 4a–c show the interaction between a dromion and a folded solitary wave for the field $|U|$ with the choices

$$\begin{aligned}
&p_x=0.8\mathrm{sech}(\xi)^2+0.1\mathrm{sech}(\xi-0.25t)^2,\\
&p=\int^{\xi}p_x x_\xi \mathrm{d}\xi=\left(-0.15-\frac{\tanh(t)}{6}\right.\\
&\left.-\frac{\tanh(t)}{12}\mathrm{sech}(\xi-0.25t)^2\right)\tanh(\xi-0.25t)\\
&+\left(-\frac{8}{15}-2\tanh(t)-\frac{2}{3}\mathrm{sech}(\xi)^2\right)\tanh(\xi)\\
&+8(\alpha-1)^{-3}\alpha(\alpha+1)\ln\frac{\alpha+\beta}{\beta+1}(2\tanh(t)+0.25)\\
&-2[(-2\alpha+1+\alpha^2)(\beta+1)(\alpha+\beta)]^{-1}[2\tanh(t)(\alpha^3\\
&+5\alpha^2\beta+6\alpha^2+2\alpha\beta+\alpha+\beta)+0.25\alpha(\alpha^2\beta+\alpha^2\\
&+2\alpha\beta+6\alpha+5\beta+1)]+8,\\
&\alpha\equiv\exp(0.5t),\quad \beta\equiv\exp(2\xi),\\
&x=\xi-2.5\tanh(\xi)-2.5\tanh(t)\tanh(\xi-0.5t),\\
&q_y=\mathrm{sech}(\eta)^2,\\
&q=\int^{\eta}q_y y_\eta \mathrm{d}\eta=\frac{14}{15}\tanh(\eta)-\frac{1}{3}\tanh(\eta)\mathrm{sech}(\eta)^2,\\
&y=\eta-0.1\tanh(\eta).
\end{aligned} \tag{39}$$

From these evolution profiles (Fig. 4) and the corresponding sectional view (Fig. 5), obviously one can observe that the interaction is nonelastic since the shapes of the interacting solitons are not preserved. There exist two multi-valued folded solitary waves after their collision, which is a novel phenomenon different from the reported cases in previous literature. The total phase shift for the static folded solitary wave in this case is

$$\begin{aligned}
\Delta_1^{+}-\Delta_1^{-}&=G_2(-\infty)R_2(-\infty)-G_2(+\infty)R_2(+\infty)\\
&=0.
\end{aligned} \tag{40}$$

5. Summary

By means of the ETM, the (2+1)-dimensional NLS equation was successfully solved. Thanks to the arbitrary functions in its solutions, we were allowed to choose them as some combinations of some exponential functions with some constant parameters. We then found a rich variety of localized excitations, such as straight-line-type solitons, curve-type solitons and pe-

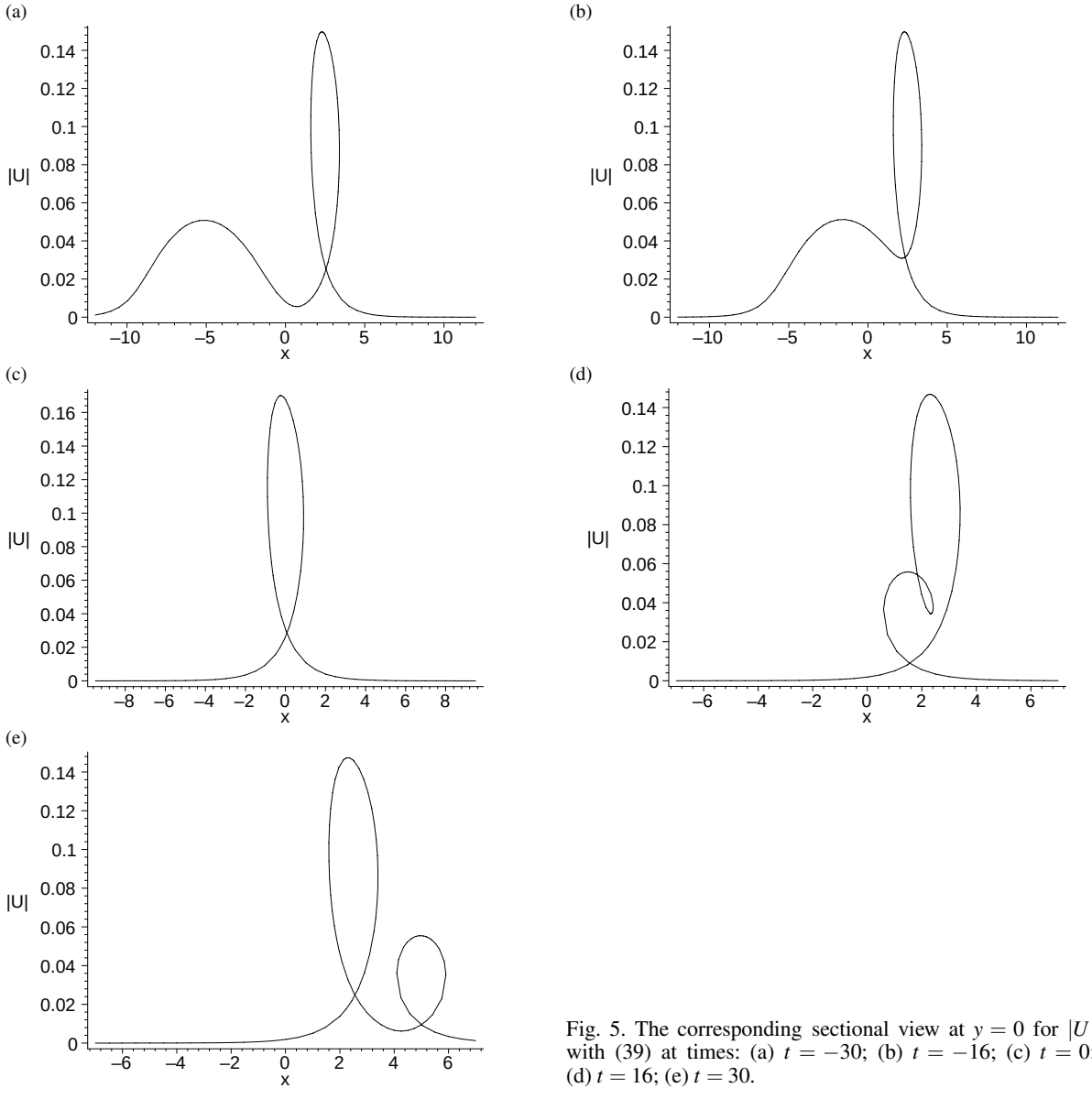

Fig. 5. The corresponding sectional view at $y = 0$ for $|U|$ with (39) at times: (a) $t = -30$; (b) $t = -16$; (c) $t = 0$; (d) $t = 16$; (e) $t = 30$.

riodic solitary waves. In addition, a dromion changed its shape after the collision with a folded solitary wave.

Because of the arbitrary functions in the solution formula of these higher-dimensional integrable models, the interactions among the localized excitations are very stimulating and are far from being an exhausted area of research. The interactions among the (2+1)-dimensional localized excitations of the NLS equation may be completely elastic or nonelastic. The interactions among the localized excitations may or may not induce phase shifts. For instance, one can get dromion reflection for $|U|$ with the following choices:

$$\begin{aligned} &p = \frac{e_0 + e_1\chi}{e_2 + e_3\chi}, \\ &\chi = \exp(k_1 x + \omega_1 t) + \exp(k_2 x + \omega_2 t), \\ &q = \exp(K_1 y) + \exp(K_2 y), \quad c_2 = 0, \end{aligned} \tag{41}$$

where

$$\begin{aligned} &e_0 = e_3 = 2.5, \quad e_1 = e_2 = k_1 = -K_1 = 1, \\ &\omega_2 = \omega_1 = 1, \quad k_2 = -K_2 = \frac{2}{5}. \end{aligned} \tag{42}$$

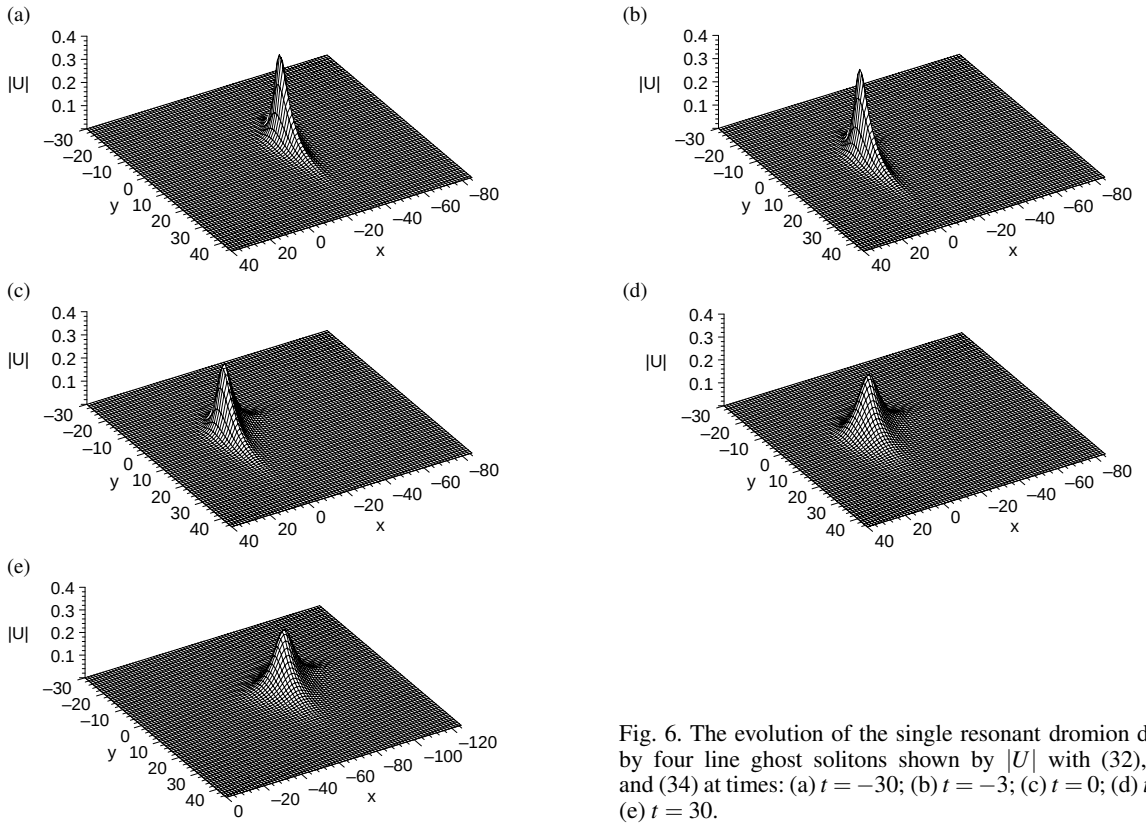

Fig. 6. The evolution of the single resonant dromion driven by four line ghost solitons shown by $|U|$ with (32), (33) and (34) at times: (a) $t=-30$; (b) $t=-3$; (c) $t=0$; (d) $t=2$; (e) $t=30$.

In this case, a dromion is reflected by an invisible ghost wall which is caused by a ghost-line soliton. The resonant dromion's shape will change after its collision with the wall. From Fig. 6, one can see that before a special 'interacting' time, the dromion moves in one direction. After the 'interacting' time, the dromion prolongs oppositely. This phenomenon resembles a moving ball reflected by a wall. Being invisible, we call it a 'ghost wall', which is caused by a ghost-line soliton. This novel interaction phenomenon is quite universal in high dimensions [19]. More about the new interactions will be studied later.

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