Exact Travelling Wave Solutions of Toda Lattice Equations Obtained via the Exp-Function Method

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We generalize the exp-function method, which was used to find new exact travelling wave solutions of nonlinear partial differential equations or coupled nonlinear partial differential equations, to nonlinear differential-difference equations. As illustration, we study two Toda lattices and obtain some new travelling wave solutions by means of the exp-function method. As some special examples, some new exact travelling wave solutions can degenerate into the kink-type solitary wave solutions reported in open literatures.

Key words: Toda Lattices; Exact Travelling Wave Solution; Exp-Function Approach.
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1. Introduction

Seeking for exact solutions of nonlinear evolution equations has flourished into a research area of great importance and interest during the last two decades. The investigation of exact solutions of nonlinear differential-difference equations (NDDEs) plays an important role in the study of nonlinear physical phenomena, and gradually becomes one of the most important and significant tasks. Recently, there has been some interest in exactly solvable discretized nonlinear models in terms of the tanh-function. Dai et al. [3] obtained the kink-type solutions of the discrete sine-Gordon equation by means of the hyperbolic function approach. The investigation of exact solutions of nonlinear partial differential equations has flourished into a research area of great importance and interest during the last two decades. The exp-function method for NDDEs step by step.

2. The Exp-Function Method for NDDEs

In this section, we would like to outline the exp-function method for NDDEs step by step. Consider a system of M polynomial NDDEs

\[
\begin{align*}
\Delta (u_{n+p_1}(x), \ldots, u_{n+p_k}(x)), \\
\frac{u_{n+p_1}(x)}{n+p_1(x)}, \ldots, \frac{u_{n+p_k}(x)}{n+p_k(x)}, \\
\frac{u^{(r)}_{n+p_1}(x)}{n+p_1(x)}, \ldots, \frac{u^{(r)}_{n+p_k}(x)}{n+p_k(x)} = 0,
\end{align*}
\]

where the dependent variable \(u_n\) has \(M\) components \(u_{n,r}\), the continuous variable \(x\) has \(N\) components \(x_i\), the discrete variable \(n\) has \(Q\) components \(n_j\), the \(k\) shift vectors \(p_i \in Z^Q\), and \(u^{(r)}(x)\) denotes the collection of mixed derivative terms of order \(r\). According to the description of the tanh-function method in [2], the main steps of the exp-function method for NDDEs are outlined as follows.

**Step 1:** When we seek the travelling wave solutions of (1), the first step is to introduce the wave transformation \(u_{n+p_i}(x) = U_{n+p_i}(\xi_n)\), \(\xi_n = \sum d_i n_i + \sum_{j=1}^N \gamma_j x_j + \zeta\) for any \(s (s = 1, \ldots, k)\), where the coefficients \(c_1, c_2, \ldots, c_N, d_1, d_2, \ldots, d_Q\) and the phase \(\zeta\)
are all constants. In this way, (1) becomes
\[
\begin{align*}
\triangle(U_{n+p_1}(ξ_n), \ldots, U_{n+p_k}(ξ_n)), \\
U_{n+p_1}(ξ_n), \ldots, U_{n+p_k}(ξ_n), \ldots, \\
U_{n+p_k}(ξ_n), \ldots, U_{n+p_k}(ξ_n)) = 0. 
\end{align*}
\] (2)

**Step 2:** We propose the following series expansion as a solution of (2):
\[
U_n(ξ_n) = \frac{\sum_{l=-p}^{q} a_l \exp(lξ_n)}{\sum_{m=-e}^{f} b_m \exp(mξ_n)}.
\] (3)

where \(p, q, e\) and \(f\) are positive integers, which are given by the homogeneous balance principle, \(a_l\) and \(b_m\) are unknown constants to be determined. To determine the values of \(f\) and \(q\), we balance the linear term of highest order in (2) with the highest-order nonlinear term. Similarly to determine the values of \(e\) and \(p\), we balance the linear term of lowest order in (2) with the lowest-order nonlinear term.

Simple computation leads to the identity
\[
ξ_{n+p_k} = ξ_n + ϕ_t,
\]
with
\[
ϕ_t = p_1d_1 + p_2d_2 + \cdots + p_qd_q.
\] (4)

Thus,
\[
U_{n+p_k}(ξ_n) = \frac{\sum_{l=-p}^{q} a_l \exp[l(ξ_n + ϕ_t)]}{\sum_{m=-e}^{f} b_m \exp[m(ξ_n + ϕ_t)]}.
\] (5)

**Remark:** Unlike difference equations which are fully discretized and differential equations which are fully continuous, NNDEs are semi-discretized with some (or all) of their spatial variables discretized while time is usually kept continuous. The key, how to apply familiar methods of the continuous case into the discrete case, is to search iterative relations between lattice indices, for example, the relations between indices \(n\) and \(n + 1\). That is to say, the key is seeking the iterative relation between (3) and (5) with (4).

**Step 3:** Determine the degrees \(p, q, e\) and \(f\) of the polynomial solutions (3) and (5). We can easily get the degrees \(p, q, e\) and \(f\) in ansatz (3) and ansatz (5) by balancing the highest nonlinear term and the highest-order derivative term in \(U_n(ξ_n)\) as in the continuous case.

3. Exact Travelling Wave Solutions of Some Toda Lattices

In this section, we apply the method developed in the preceding section to some Toda lattices.

3.1. Example 1

The Toda lattice [15] is of the form
\[
\frac{d^2u_n}{dr^2} = \left(\frac{du_n}{dr}\right)^2 + (u_{n-1} - 2u_n + u_{n+1}).
\] (6)

In this case, \(u = u, x = \{x_1, x_2\} = \{t, x\}, n = n_1 = n\) and \(p_1 = p_1 = -1, p_2 = p_2 = 0, p_3 = p_3 = 1, d_1 = d, c_1 = c\).

Now we assume that (6) has the solutions as ansatz (3). The homogeneous balance procedure in Step 2 of Section 2 leads to the results \(q = f, p = e\). We can freely choose the values of \(p\) and \(q\), but the final solution does not strongly depend on the choice of values of \(p\) and \(q\) [12, 13]. For simplicity, we set \(q = f = 1\) and \(p = e = 1\). Thus we give the following formal travelling wave solutions of (6):

\[
\begin{align*}
u_n &= a_{-1}\exp(-ξ_n) + a_0 + a_1\exp(ξ_n), \\
u_{n+1} &= a_{-1}\exp(-ξ_{n+1}) + b_0 + b_1\exp(ξ_{n+1}), \\
u_{n+1} &= a_{-1}\exp(-ξ_{n+1} - d) + b_0 + b_1\exp(ξ_{n+1} + d), \\
u_{n-1} &= a_{-1}\exp(-ξ_{n-1} + d) + b_0 + b_1\exp(ξ_{n-1} - d),
\end{align*}
\] (7)

with
\[
ξ_n = dn + ct + ζ,
\] (8)

where \(a_0, a_1, a_{-1}, b_0, b_1, b_{-1}, d\) and \(c\) are constants to be determined later. Inserting the expressions (7) and...
(8) into (6), clearing the denominator and eliminating the coefficients of independent terms in \(\exp(j\xi_n)\) \((j = 1, 2, \ldots)\), yields a system of algebraic equations, from which we obtain many travelling wave solutions of the Toda lattice equation (6). Here we only partially list these solutions.

**Case 1.**

\[
a_0 = b_0 = 0, \\
a_{-1} = \left[\frac{a_1}{b_1} \mp 2 \sinh(d)\right] b_{-1}, \\
c = \pm \sinh(d),
\]

where \(a_1, b_1, b_{-1}\) and \(d\) are arbitrary constants. From (7)–(9), we can obtain the exact travelling wave solution of the Toda lattice equation (6), i.e.,

\[
u_n = \frac{1}{b_{-1}} \exp(-\xi_n) + a_1 \exp(\xi_n),
\]

where \(\xi_n = dn \pm \sinh(d)\) \(t + \xi\). As a special example, when \(a_1 = A_0 \pm \sinh(d), b_{-1} = b_1 = 1\), the solution (10) is

\[
u_n = A_0 \pm \sinh(d) \tanh(\xi_n),
\]

which turns out to be Baldwin’s [2] and Dai’s [16] kink-type solitary wave solution as expressed in (11).

As another special example, when \(a_1 = A_0 \pm \sinh(d), b_{-1} = b_1 = -1\), the solution (10) is

\[
u_n = A_0 \pm \sinh(d) \coth(\xi_n),
\]

which turns out to be Dai’s [16] travelling wave solution as expressed in (12).

**Case 2.**

\[
a_{-1} = b_{-1} = 0, \\
a_1 = b_1 \left[\frac{a_0}{b_0} \pm 2 \sinh(d)\right], \\
c = \pm 2 \sinh(d),
\]

where \(a_0, b_0, b_1\) and \(d\) are arbitrary constants. From (7), (8) and (13), we can obtain the exact travelling wave solution of the Toda lattice equation (6), i.e.,

\[
u_n = \frac{a_0 + b_1 \frac{a_0}{b_0} \pm 2 \sinh(d)}{b_0 + b_1 \exp(\xi_n)} \exp(\xi_n),
\]

where \(\xi_n = dn \pm 2 \sinh(\xi_n) t + \xi\).

**Case 3.**

\[
a_1 = b_1 = 0, \\
a_0 = b_0 \left[\frac{a_{-1}}{b_{-1}} \pm 2 \sinh\left(\frac{d}{2}\right)\right], \\
c = \pm 2 \sinh\left(\frac{d}{2}\right),
\]

where \(b_0, a_{-1}, b_{-1}\) and \(d\) are arbitrary constants. From (7), (8) and (15), we can obtain the exact travelling wave solution of the Toda lattice equation (6), i.e.,

\[
u_n = \frac{a_{-1} \exp(-\xi_n) + b_0 \left[\frac{a_{-1}}{b_{-1}} \pm 2 \sinh\left(\frac{d}{2}\right)\right]}{b_{-1} \exp(-\xi_n) + b_0},
\]

where \(\xi_n = dn \pm 2 \sinh(\xi_n) t + \xi\).

**Case 4.**

\[
b_1 = \frac{b_1^2}{4b_{-1}}, \\
a_0 = \pm \sinh\left(\frac{d}{2}\right) b_0 + \frac{a_{-1} b_0}{b_{-1}}, \\
a_1 = \pm \frac{b_1^2}{2b_{-1}} \sinh\left(\frac{d}{2}\right) + \frac{a_{-1} b_0^2}{4b_{-1}^2}, \\
c = \pm 2 \sinh\left(\frac{d}{2}\right),
\]

where \(a_{-1}, b_0, b_{-1}\) and \(d\) are arbitrary constants. From (7), (8) and (17), we can obtain a new exact travelling wave solution of the Toda lattice equation (6), i.e.,

\[
u_n = \left\{4a_{-1} b_{-1}^2 \exp(-\xi_n) + 4a_{-1} b_0 b_{-1} \right. \\
\left. \pm 4b_0 b_{-1}^2 \sinh\left(\frac{d}{2}\right) \exp(-\xi_n) \right. \\
\left. + \left[a_{-1} b_0^2 \pm 2b_0^2 b_{-1} \sinh\left(\frac{d}{2}\right) \exp(-\xi_n) \right] \exp(-\xi_n) \right. \\
\left. - 1 \right\},
\]

where \(\xi_n = dn \pm 2 \sinh(\xi_n) t + \xi\).

It is well known that soliton solutions are interesting and physical relevant. Here we take the solution (10) as example to further analyze its properties by some figures. The wave velocity “c” of the solution (10) changes with wave number “d” plotted in Figure 1a. The solution (10) is a kink-type solitary wave shown in Fig. 1b by selecting appropriate parameters. As can be seen from Fig. 1b, with the growth of the
wave number, the distance between the lattices becomes gradually bigger in the kink location of wave. Meanwhile, the asymptotic properties of the solution (10) with different lattice labels \( n \) are plotted in Figure 1c. Besides the property of the kink-type solitary wave, the solution (10) exhibits the feature of an exotic wave shown in Figure 1d. The mathematical expression of this exotic wave has the similar form as (12). However, different from the continuous case, the exotic wave doesn’t exhibit singular properties although it is composed of the singular coth-function. When the parameters \( b_1 b_{-1} > 0 \) and \( b_1 b_{-1} < 0 \), the solution (10) is a kink-type solitary wave and an exotic wave, respectively.

3.2. Example 2

Another Toda lattice [17] has the form

\[
\frac{du_n}{dt} = u_n (v_n - v_{n-1}), \quad \frac{dv_n}{dt} = v_n (u_{n+1} - u_n). \tag{19}
\]

In this case, \( u = \{u, v\} \), \( x = x_1 = t \), \( n = n_1 = n \) and \( p_1 = p_1 = -1, p_2 = p_2 = 0, p_3 = p_3 = 1 \).

The balance procedure admits us to give travelling wave solutions of (19) and, in case of \( b_1 \neq 0, d_1 \neq 0 \), this ansatz can be simplified as

\[
u_n = -a \exp(-\xi_n) + a_0 + a_1 \exp(\xi_n), \\
\mu_{n+1} = a \exp(-\xi_n - d) + a_0 + a_1 \exp(\xi_n + d), \tag{20}
\]

\[
u_{n-1} = b \exp(-\xi_n - d) + b_0 + \exp(\xi_n + d), \\
\mu_{n-1} = c \exp(-\xi_n + d) + c_0 + c_1 \exp(\xi_n - d), \tag{21}
\]

\[
u_n = d \exp(-\xi_n) + d_0 + \exp(\xi_n), \\
\mu_{n+1} = e \exp(-\xi_n - d) + e_0 + \exp(\xi_n + d), \tag{22}
\]

with \( \xi_n = dn + ct + \zeta \).

Substituting ansatz (20) and ansatz (21) with (22) into (19), clearing the denominator and eliminating all...
the coefficients of the powers of $\exp(j \xi_n)$ ($j = 1, 2, \ldots$) yields a nonlinear algebraic system. Proceeding as before we only partly list these solutions.

**Case 1.**

\[
\begin{align*}
a_0 &= b_0 = c_0 = d_0 = 0, \\
a_{-1} &= \exp(2d)c_{-1}, & a_1 &= -c \exp(-d)\csc(d), \\
b_{-1} &= d_{-1} = -\frac{c_{-1}}{c} \exp(d) \sinh(d), \\
c_1 &= -c \exp(d) \csc(d),
\end{align*}
\]

(23)

where $c_{-1}, d$ and $c$ are arbitrary constants. From (20)–(23) we can obtain exact travelling wave solutions of the Toda lattice equation (19), i.e.,

\[
\begin{align*}
u_n &= \frac{\exp(2d)c_{-1} \exp(-\xi_n) - c^2 \exp(-d) \csc(d) \exp(\xi_n)}{-c_{-1} \exp(d) \sinh(d) \exp(-\xi_n) + c \exp(\xi_n)}, \\
u_n &= \frac{c_{-1} \exp(-\xi_n) - c^2 \exp(d) \csc(d) \exp(\xi_n)}{-c_{-1} \exp(d) \sinh(d) \exp(-\xi_n) + c \exp(\xi_n)},
\end{align*}
\]

(24)

where $\xi_n = dn + ct + \zeta$.

**Case 2.**

\[
\begin{align*}
a_0 &= b_0 = c_0 = d_0 = 0, \\
a_{-1} &= c_{-1}, \\
a_1 &= -c \exp(d) \csc(d), \\
b_{-1} &= -\frac{c_{-1}}{c} \exp(d) \sinh(d), \\
d_{-1} &= -\frac{c_{-1}}{c} \exp(-d) \sinh(d), \\
c_1 &= -c \exp(-d) \csc(d),
\end{align*}
\]

(25)

where $c_{-1}, d$ and $c$ are arbitrary constants. From (20)–(22) and (25) we can obtain exact travelling wave solutions of the Toda lattice equation (19), i.e.,

\[
\begin{align*}
u_n &= \frac{c_{-1} \exp(-\xi_n) - c^2 \exp(d) \csc(d) \exp(\xi_n)}{-c_{-1} \exp(d) \sinh(d) \exp(-\xi_n) + c \exp(\xi_n)}, \\
u_n &= \frac{c_{-1} \exp(-\xi_n) - c^2 \exp(-d) \csc(d) \exp(\xi_n)}{-c_{-1} \exp(-d) \sinh(d) \exp(-\xi_n) + c \exp(\xi_n)},
\end{align*}
\]

(26)

where $\xi_n = dn + ct + \zeta$.

**Case 3.**

\[
\begin{align*}
a_{-1} &= \frac{cd_0^2}{8} \exp(d) \csc(2d) \csc^2 \left( \frac{d}{2} \right), \\
b_{-1} &= \frac{d_0^2}{4} \exp(d) \csc^2(d), \\
c_{-1} &= 4cd_0^2 \csc^3(d) \cosh(d), \\
d_{-1} &= \frac{d_0^2}{4} \csc^2(d), & c_0 &= 0,
\end{align*}
\]

(27)

\[
\begin{align*}
a_0 &= -cd_0 \exp \left( \frac{d}{2} \right) \csc \left( \frac{d}{2} \right) \left[ 1 + \frac{1}{2} \csc(d) \right], \\
b_0 &= \frac{d_0}{2} \exp \left( \frac{d}{2} \right) \csc \left( \frac{d}{2} \right), \\
a_1 &= \frac{c}{2} \coth \left( \frac{d}{2} \right) \csc(d), \\
c_1 &= -c \coth(d),
\end{align*}
\]

where $d_0, c$ and $d$ are arbitrary constants. From (20)–(22) and (27) we can obtain new exact travelling wave solutions of the Toda lattice equation (19), i.e.,

\[
\begin{align*}
u_n &= \left\{ \frac{cd_0^2}{8} \exp(d) \csc(2d) \csc^2 \left( \frac{d}{2} \right) \exp(-\xi_n) \\
&\quad - cd_0 \exp \left( \frac{d}{2} \right) \csc \left( \frac{d}{2} \right) \left[ 8 + 4 \csc(d) \right] \\
&\quad + 4c \coth \left( \frac{d}{2} \right) \csc(d) \exp(\xi_n) \right\},
\end{align*}
\]

\[
\begin{align*}
\nu_n &= \left\{ 2d_0^2 \exp(d) \csc^2(d) \exp(-\xi_n) \\
&\quad + 4d_0 \exp \left( \frac{d}{2} \right) \csc \left( \frac{d}{2} \right) + 8 \exp(\xi_n) \right\}^{-1},
\end{align*}
\]

(28)

\[
\begin{align*}
u_n &= \left\{ 16cd_0^2 \csc^3(d) \cosh(d) \exp(-\xi_n) \\
&\quad - 4c \coth(d) \exp(\xi_n) \right\},
\end{align*}
\]

\[
\begin{align*}
\nu_n &= \left\{ d_0^2 \csc^2(d) \exp(-\xi_n) + 4d_0 + 4 \exp(\xi_n) \right\}^{-1},
\end{align*}
\]

where $\xi_n = dn + ct + \zeta$.

Figure 2a displays the solitary wave property of solution (24). From Fig. 2a, the physical quantity $u$ is an kink-type solitary wave (squares), and $v$ is an anti-kink solitary wave (circles). Similarly to solution (10), the solution (24) exhibits also the feature of an exotic wave as shown in Figure 2b. When $cdc_{-1} < 0$ and $cdc_{-1} > 0$, the solutions of (24) are kink-type solitary waves and exotic wave waves, respectively.
4. Summary and Discussion

In summary, we systematically illustrated the solution procedure of the exp-function method for nonlinear differential-difference equations. The solution procedure is very simple, and the obtained solution is very concise. We obtained some new exact travelling wave solutions of the Toda lattices via the exp-function method. As some special examples, these new exact travelling wave solutions can degenerate into the kink-type solitary wave solutions reported in [2, 16]. So using the proposed exp-function method one can obtain easily travelling wave solutions and kink-type solitary wave solutions of nonlinear differential-difference equations. Of course, lacking theory and experiments related to these solutions of the Toda lattices, we could not further say something about the real physical meaning of our exact solutions. Although these solutions are only a small part of the large variety of possible solutions for the equations discussed here, they might serve as seeding solutions for a class of localized structures which exist in these systems. We hope that they will be useful in further perturbative and numerical analysis of various solutions these lattice equations.

The presented method is only an initial work, more work will be done. Additional applications of this method to other nonlinear differential-difference systems deserve further investigation.