Reconstruction of the Potential Function and its Derivatives for the Diffusion Operator

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We solve the inverse nodal problem for the diffusion operator. In particular, we obtain a reconstruction of the potential function and its derivatives using only nodal data. Results are a generalization of Law’s and Yang’s works.

Key words: Diffusion Operator; Inverse Nodal Problem; Reconstruction Formula.

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1. Introduction

The inverse nodal problem was initiated by McLaughlin [1], who proved that the Sturm-Liouville problem is uniquely determined by any dense subset of the nodal points. Some numerical schemes were given by Hald and McLaughlin [2] for the reconstruction of the potential. Recently Law, Yang and other authors have reconstructed the potential function and its derivatives of the Sturm-Liouville problem from the nodal points [3 – 7].

In this paper, we are concerned with the inverse nodal problem for the diffusion operator on a finite interval. We reconstruct the potential function and all its derivatives by using Law’s and Yang’s method [7].

The diffusion operator is written as

\[ L_y = -y'' + [g(x) + 2\lambda p(x)]y, \]

where the function \( g(x) \in L^2[0, \pi], \) \( p(x) \in L^2[0, \pi], \) Some spectral problems were extensively solved for the diffusion operator in [8 – 11].

Consider the problem

\[ L[y] = \lambda^2 y, \]

\[ y(0) = 1, \quad y'(\pi, \lambda) + Hy(\pi, \lambda) = 0, \]

where \( \lambda \) and \( H \) are finite numbers.

Let \( \lambda_n \) be the \( n \)-th eigenvalue and \( 0 < x_0^n < \ldots < x_i^n < \pi, i = 1, 2, \ldots, n - 1 \), the nodal points of the \( n \)-th eigenfunction. Also let \( I^n_i = [x_i^n, x_{i+1}^n] \) be the \( i \)-th nodal domain of the \( n \)-th eigenfunction and let \( I^n_n = |I^n_0| = x_{n+1}^n - x_0^n \) be the associated nodal length. Let \( j_n(x) \) be the largest index \( j \) such that \( 0 \leq x_j^n < x \).

\( \Delta \) denotes the difference operator \( \Delta a_i = a_{i+1} - a_i. \)

Inductively, for \( k > 1, \Delta^k a_i = \Delta^{k-1} a_{i+1} - \Delta^{k-1} a_i, \) and we introduce the difference quotient operator \( \delta: \)

\[ \delta a_i = \frac{a_{i+1} - a_i}{x_{i+1} - x_i} = \frac{\Delta a_i}{l_i} \]

and \( \delta^k a_i = \frac{\delta^{k-1} a_{i+1} - \delta^{k-1} a_i}{l_i}. \)

2. Main Results

**Lemma 1.** [12] Assume that \( q \in L^2[0, \pi]. \) Then, as \( n \to \infty \) for the problem (1.2) – (1.4),

\[ x_i^n = \frac{(i - \frac{1}{2}) \pi}{\lambda_n} - \frac{h}{2\lambda_n^2} + \frac{1}{2\lambda_n^2} \int_0^{x_i^n} \left( 1 + \cos 2\lambda_n t \right) q(t) + 2\lambda_n p(t) \] \( \)dt + \( O \left( \frac{1}{\lambda_n^3} \right) \)

\[ l_i^n = \frac{\pi}{\lambda_n} + \frac{1}{2\lambda_n^2} \int_{x_{i-1}^n}^{x_i^n} \left( 1 + \cos 2\lambda_n t \right) q(t) \]

\( + 2\lambda_n p(t) \)dt + \( O \left( \frac{1}{\lambda_n^3} \right) \).
Lemma 2. Suppose that $f \in L^2[0, \pi]$. Then for almost every $x \in [0, \pi]$, with $j = j_n(x)$,
\[
\lim_{n \to \infty} \frac{\lambda_n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} f(t) dt = f(x).
\]

Theorem 1. [12] Suppose that $q \in L^2[0, \pi]$, then
\[
q(x) = \lim_{n \to \infty} \left( \frac{2\lambda_n \pi}{2 - 2\lambda_n - 2p(x)} \right).
\]

Proposition 1. If $q$ is a continuous function, then
(a) $\lim_{n \to \infty} \sqrt{\lambda_n} l_j^{(n)} = \pi$ and $l_j^{(n)} = \frac{1}{n} + O \left( \frac{1}{n^2} \right)$;
(b) $l_j^{(n)} f_j^{(n)} = 1 + O \left( \frac{1}{n} \right)$ for any fixed $k, m \in N$;
(c) $q_m \leq \lambda_m - \frac{\pi^2}{(l_j^{(n)})^2} \leq q_M$, where $q_m = \min_{[0, \pi]} q(x)$ and $q_M = \max_{[0, \pi]} q(x)$.

Lemma 3. [7] If $q \in C^N[0, \pi]$, then for $k = 1, \ldots, N, \Delta^k l_j = O(n^{-(k+3)})$ as $n \to \infty$ and the order estimate is independent of $j$.

Lemma 4. [7] Let $\Phi_j = \sum_{i=1}^{m} \phi_{j,i}$ with each $\phi_{j,i} = \prod_{p=1}^{k} \varphi_{j,i,p}$, where each $\varphi_{j,i,p} \in U_{j}^{(n)}$. Suppose $\Phi_j = O(n^{-q})$ and $q$ is sufficiently smooth. Then $\delta^k \Phi_j = O(n^{-q})$ for all $k \in N$.

Lemma 5. [7] Suppose $f \in C^N[0, \pi]$ and $\Phi_j = \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} f(x) dx$. Then $\delta^k \Phi_j = O(n^{-q})$ for any $k = 0, 1, \ldots, N$.

Theorem 2. [7] Let $\Phi_m(x_j) = \psi_1(x_j) \psi_2(x_j) \cdots \psi_m(x_j)$, where $\psi_i(x_j) = x_{j+k_i}$ and $k_i \in N \cup \{0\}$. If $f$ is $C^k$ on $[0, \pi]$, then
\[
\delta^k \Phi_m(x_j) = \begin{cases} O(1), & 0 \leq k \leq m-1, \\ m! + O(n^{-1}), & k = m, \\ O(n^{-2}), & k \geq m+1. \end{cases}
\]

Theorem 3. If $q \in C^{k+1}[0, \pi]$, then $q^{(k)}(x) = \delta^k q(x_j) - 2\lambda_n p(x) + O(n^{-1})$ for $k = 0, 1, \ldots, N$, where $j = j_m(x)$. The order estimate is uniformly valid for compact subsets of $[0, \pi]$.

Remark: For $k = 1, \ldots, N$,
\[
V_k(x_j) = \begin{bmatrix} 1 & x_j & \cdots & x_j^{k-1} \\ 1 & x_{j+1} & \cdots & x_{j+1}^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{j+k} & \cdots & x_{j+k}^{k-1} \end{bmatrix},
\]
be a $(k+1) \times (k+1)$ Vandermonde matrix. It is well known that
\[
\det V_k(x_j) = \prod_{m=1}^{k} \prod_{i=0}^{m-1} \left( \sum_{p=1}^{l_{j+k-1}} \right).
\]

To prove Theorem 3, we need the following lemma.

Lemma 6.
\[
\prod_{m=1}^{k} \prod_{i=0}^{m-1} \left( \sum_{p=1}^{l_{j+k-1}} \right) + O \left( \frac{1}{n^2} \right).
\]

Next, we consider the following $(k+1) \times (k+1)$ matrix:
\[
A = \begin{bmatrix} 1 & x_j & \cdots & x_j^{k-1} & q(x_j) \\ 1 & x_{j+1} & \cdots & x_{j+1}^{k-1} & q(x_{j+1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_{j+k} & \cdots & x_{j+k}^{k-1} & q(x_{j+k}) \end{bmatrix}.
\]

After some operations, we obtain
\[
\det A = l_{j+1}^{(1)} \cdots l_{j+k-1}^{(k)} - \dfrac{\delta^k q(x_j) + \dfrac{1}{m!} \delta^k q(x_j)}{\delta^k q(x_j) + \dfrac{1}{m!} \delta^k q(x_j)}.
\]

Let $B$ be the matrix at the right-hand side. By Lemma 4 and Theorem 2,
\[
\det B = (\delta^k q(x_j))^{(k-1)} \cdots \delta^k q(x_j) + O \left( \frac{1}{n^2} \right).
\]
**Proof of Theorem 3:** For \( k = 1, 2, \ldots, N \), let
\[
g(x) = \det \begin{bmatrix} 1 & x & \cdots & x^k & q(x) \\ 1 & x_j & \cdots & x_j^k & q(x_j) \\ \vdots & \vdots & & \vdots \\ 1 & x_{j+k} & \cdots & x_{j+k}^k & q(x_{j+k}) \end{bmatrix}.
\]
By Rolle’s theorem, \( g(x_j) = g(x_{j+1}) = \ldots = g(x_{j+k}) = 0 \) implies that there is some \( x_{i,j+i} \in (x_{j+i}, x_{j+i+1}) \) such that \( g'(x_{i,j+i}) = 0 \). When we repeat the process, we can find that \( x_{k,j} \in (x_j, x_{j+k}) \) such that \( g^{(k)}(x_{k,j}) = 0 \). In view of the definition of \( g \), \( g^{(k)}(x_{k,j}) \) is equal to \( k! \det A \). Hence
\[
g^{(k)}(x_{k,j}) = (k!) (l_j)^k (l_{j+1})^{k-1} \cdots l_{j+k-1} \frac{\det B}{\det V_k(x_j)}.
\]
By Lemma 6, \( g^{(k)}(x_{k,j}) = \delta^k q(x_j) + O(\frac{1}{n}) \), since \( \delta^k q(x_j) = g^{(k)}(x_{k,j}) - 2\lambda_n p(x) + O(\frac{1}{n}) \).

**Theorem 4.** Suppose that \( q \) in (1.1) is \( C^N \) on \([0, \pi]\) \((N \geq 1)\), and let \( j = j(x) \) for each \( x \in [0, \pi] \).
Then, as \( n \to \infty \),
\[
q(x) = \lambda_n \left( \frac{2\lambda_n^2 p}{2\lambda_n - 2p(x)} \right) + O\left( \frac{1}{n} \right),
\]
and, for all \( k = 1, 2, \ldots, N \),
\[
q^{(k)}(x) = \frac{2\lambda_n^3/2 \delta^k l_j}{2\lambda_n - 2\lambda_n \delta^k p(x)} - 2\lambda_n \delta^k p(x) + O(1).
\]

**Proof:** The uniform approximation for \( q \) is evident. Suppose that \( q \) is continuously differentiable on \([0, \pi]\). Apply the intermediate value theorem on Proposition 1c, then there is some \( \xi_{j,n} \in (x_j, x_{j+1}) \) such that
\[
\lambda_j^2 q(n) = \left( 1 - \frac{q(n)}{\lambda_n} \right)^{-1/2} = 1 + \frac{q(n)}{2\lambda_n} + O\left( \frac{1}{n^2} \right).
\]
Hence
\[
2\lambda_n \left( \frac{\lambda_n^2 p}{\lambda_n} - 1 \right) - q(n) = O\left( \frac{1}{n^2} \right),
\]
\[
2\lambda_n \left( \frac{\lambda_n^2 p}{\lambda_n} - p(x) \right) + 2\lambda_n p(x) - q(n) = O\left( \frac{1}{n^2} \right).
\]
Applying the mean value theorem, when \( n \) is sufficiently large, then
\[
q(x) = q(x_n) - 2\lambda_n p(x) + O\left( \frac{1}{n} \right).
\]
Then we employ a modified Prüfer substitution due to Ashbaugh-Benguria [13] to solve the boundary conditions \( h \) and \( H, x = 0 \) and \( x = \pi \), respectively:
\[
\begin{align*}
y &= r(x) \sin \sqrt{\lambda} \theta(x), \\
y' &= \sqrt{\lambda} r(x) \cos \sqrt{\lambda} \theta(x),
\end{align*}
\]
so that
\[
\theta' = \cos^2 \sqrt{\lambda} \theta(x) - \frac{q(x) \sin^2 \sqrt{\lambda} \theta(x)}{\sqrt{\lambda}} - 2p(x) \sin^2 \sqrt{\lambda} \theta(x). \tag{2.4}
\]
Integrating (2.4) from \( x_j \) to \( x_{j+1} \),
\[
\int_{x_j}^{x_{j+1}} \frac{\pi}{\sqrt{\lambda_n}} = \int_{x_j}^{x_{j+1}} \cos^2 \sqrt{\lambda} \theta(x) \, dx + \frac{1}{\lambda} \int_{x_j}^{x_{j+1}} q(x) \sin^2 \sqrt{\lambda} \theta(x) \, dx - \frac{1}{\lambda} \int_{x_j}^{x_{j+1}} p(x) \sin^2 \sqrt{\lambda} \theta(x) \, dx \tag{2.5}
\]
it results by Lemma 5 from (2.5) that
\[
\int_{x_j}^{x_{j+1}} \frac{\pi}{\sqrt{\lambda_n}} = - \frac{q(x_j)}{2\lambda_n} l_j + c_j + O\left( \frac{1}{n^{N+3}} \right) - p(x_j) l_j + d_j + O\left( \frac{1}{n^{N+4}} \right) + O\left( \frac{1}{n} \right), \tag{2.6}
\]
where
\[
c_j = \frac{1}{2\lambda_n} \sum_{k=1}^{N} \frac{q^{(k)}(x_j)}{(k+1)!} p^{(k)} + O\left( \frac{1}{n^3} \right),
\]
\[
d_j = \sum_{k=1}^{N} \frac{p^{(k)}(x_j)}{(k+1)!} p^{(k+1)} = O\left( \frac{1}{n^2} \right).
\]
Summarizing from (2.6),
\[
q(x_j) = - \frac{2\pi}{l_j} - 2\lambda_n p(x_j) + 2\lambda_n (c_j + d_j) + O(1).
\]
Therefore
\[ \delta q(x_j) = -2\pi \sqrt{\lambda_n} \frac{\Delta l_j}{l_j^2 l_{j+1}} - 2\lambda_n \delta p(x_j) + O(1), \]
and so, for \( k = 1, 2, \ldots, N, \)
\[ \delta^k q(x_j) = -2\pi \sqrt{\lambda_n} \delta^{k-1} \left( \frac{\Delta l_j}{l_j^2 l_{j+1}} \right) 
- 2\lambda_n \delta^k p(x_j) + O(1). \] (2.7)

If we use the results of Theorem 3 and Theorem 5, we get
\[ q^{(k)}(x) = \delta^{k} q(x_j) - 2\lambda_n p(x) + O(n^{-1}), \]
\[ q^{(k)}(x) = \frac{2\lambda_n^{3/2} \delta^k l_j}{\pi} - 2\lambda_n \delta^k p(x_j) - 2\lambda_n p(x) + O(1). \]

Theorem 5. Assume that \( q \) is \( C^{N+1} \) on \([0, \pi]\). Then, for \( k = 1, 2, \ldots, N, \)
\[ \delta^k q(x_j) = \frac{2\lambda_n^{3/2} \delta^k l_j}{\pi} - 2\lambda_n \delta^k p(x_j) + O(1). \]
The estimate is independent of \( j. \)

**Proof:** In view of derivations and the fact that \( \delta l_j = \frac{\Delta l_j}{l_j^2 l_{j+1}} = O(n^{-3}) \), it suffices to show that
\[ \delta^{k-1} \left( \frac{\pi^2 \Delta l_j}{l_j^2 l_{j+1}} \right) = -\delta^{k-1} \left( \frac{\lambda_n \Delta l_j}{l_j} \right) + O \left( \frac{1}{n^3} \right). \] (2.8)

But
\[ \delta \left( \frac{\lambda_n \Delta l_j}{l_j} \right) = -\frac{\pi^2 \Delta l_j}{l_j^2 l_{j+1}} \]
\[ = \pi^2 \left( \frac{\Delta l_j + \Delta l_{j+1}}{l_j l_{j+1} l_{j+2}} \right) + \left( \frac{\lambda_n - \pi^2}{l_j l_{j+1}} \right) \delta l_j 
= O \left( \frac{1}{n^3} \right). \]

Thus, (2.8) follows by Lemma 4. If we write (2.8) in (2.7), then
\[ \delta^k q(x_j) = -2\pi \sqrt{\lambda_n} \left[ -\delta^{k-1} \left( \frac{\lambda_n \Delta l_j}{\pi^2 l_j} \right) + O \left( \frac{1}{n^3} \right) \right] 
- 2\lambda_n \delta^k p(x_j) + O(1), \] (2.9)
\[ \delta^k q(x_j) = \frac{2\lambda_n^{3/2} \delta^k l_j}{\pi} - 2\lambda_n \delta^k p(x_j) + O(1). \]