A Classical Model for the Extended Electron I

Dietrich Lortz*

Max-Planck-Institut für Plasmaphysik, D-85748 Garching
* Retired.

Reprint requests to Prof. D. L.; E-mail:xxl@ipp-garching.mpg.de

Z. Naturforsch. **60a,** 149 – 157 (2005); received August 26, 2004

In order to avoid divergence of its self-energy the free electron at rest is described as a relativistic continuum of finite extent. It has the form of an axisymmetric torus of finite aspect ratio, which rigidly rotates around its axis of symmetry with superluminal speed $(v > c)$. It is shown that there is a class of stationary solutions of the free-boundary value problem. The parameter dependence of these solutions is related to experimental data.

Key words: Classical Field Theory; Relativistic Fluid Dynamics.

1. Introduction

In contrast to point charge theories, where the electron is described as a mathematical point with charge, mass, spin angular momentum, and electromagnetic field, the extended charge theories try to avoid the divergent self-energy by considering particles of finite extent. Not only is this much more complicated, it also needs additional terms to compensate for the Coulomb repulsion, giving rise to an unstable electron. As early as 1905 Poincare [1] suggested nonelectromagnetic cohesive forces, a kind of negative pressure. A history of many such attempts can be found in the book of Rohrlich [2] with its many references. The main difficulty of the extended charge theories is that the boundary conditions imply a so-called virial theorem. This is derived by multiplying the time-independent equation of motion by the position vector and integrating over the whole space. The boundary conditions that all nontrivial fields tend to zero at infinity then yield a contradiction. For the simplest case this was shown in [3]. For the reader's convenience the definitions and formulae are repeated here.

The electron is described in terms of a model of relativistic continuum mechanics and vacuum electrodynamics, which means that ϵ and μ have the vacuum values ϵ_0 and μ_0, respectively. The equations of vacuum electrodynamics are written in SI units:

$$\operatorname{div}\vec{B} = 0, \tag{1}$$

$$\operatorname{curl}\vec{E} + \partial_t\vec{B} = 0, \tag{2}$$

$$\operatorname{div}\vec{E} = \frac{q}{\epsilon_0}, \tag{3}$$

$$\mu_0\vec{j} = \operatorname{curl}\vec{B} - \frac{1}{c^2}\partial_t\vec{E}, \tag{4}$$

$$\vec{j} = q\vec{v}. \tag{5}$$

Here, q is the charge density, ϵ_0 and μ_0 are the capacitivity and permeability of free space, $c = (\epsilon_0\mu_0)^{-\frac{1}{2}}$ is the vacuum speed of light, and the other symbols have their usual meaning.

With the metric tensor

$$g_{\nu\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6}$$

the 4-velocity and the gradient have the form

$$u^\nu = \gamma\begin{pmatrix} c \\ v_i \end{pmatrix}, \quad \gamma = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}, \quad \partial_\nu = \begin{pmatrix} \frac{1}{c}\partial_t \\ \partial_i \end{pmatrix},$$
$$\nu, \mu = 0, 1, 2, 3, \quad i, j, k = 1, 2, 3, \tag{7}$$

and the electromagnetic field is written as

$$F^{\nu\mu} = \begin{pmatrix} 0 & -\frac{E_j}{c} \\ \frac{E_i}{c} & -B_{ij} \end{pmatrix},$$
$$B_{ij} = \epsilon_{ijk}B_k, \; j_\mu = \begin{pmatrix} cq \\ -j_i \end{pmatrix}. \tag{8}$$

In continuum mechanics with electromagnetic force the simplest relativistic forms of mass and energy-momentum balance are

$$\partial_\nu \rho_0 u^\nu = 0, \tag{9}$$

$$\rho_0 u^\mu \partial_\mu u^\nu = F^{\nu\mu} j_\mu. \tag{10}$$

Here, the mass density ρ_0 is a Lorentz invariant which is not necessarily the rest mass density. While the covariant four-vector formulation of the Maxwell equations is well known, the covariant form

$$j^\nu = u^\nu j^\mu u_\mu / c^2 \tag{11}$$

of the convection current (5) is not so familiar. Without electric conductivity (11) is the covariant form of Ohm's law. The spatial part of (11) is (5).

1.1. The Model

Equations (9) and (10) yield in 3-vector notation

$$\partial_t \gamma\rho_0 + \operatorname{div} \gamma\rho_0 \vec{v} = 0, \tag{12}$$

$$c^2 \gamma\rho_0 \left[\partial_t \gamma + (\vec{v}\cdot\nabla)\,\gamma\right] = \vec{E}\cdot\vec{\jmath}, \tag{13}$$

$$\gamma\rho_0 \left[\partial_t \gamma\vec{v} + (\vec{v}\cdot\nabla)\,\gamma\vec{v}\right] = q\vec{E} + \vec{\jmath}\times\vec{B}, \tag{14}$$

where (13) can be omitted because it is a consequence of (14).

The theory starts with Maxwell's equations (1) – (5) and (12) and (14). In the following the time-independent (stationary) case is considered. By using (3), (4), and (12) the momentum equation (14) can be written in terms of the stress tensor as [4]

$$\partial_j \left[\rho_0 \gamma^2 v_i v_j - \epsilon_0 (E_i E_j - \frac{1}{2} E^2 \delta_{ij}) - \frac{1}{\mu_0}(B_i B_j - \frac{1}{2} B^2 \delta_{ij})\right] = 0. \tag{15}$$

It has been shown in [3] that, if all fields tend to zero at infinity, there is no non-trivial real solution of (15). However, this conclusion uses the reality of γ, which implies $v < c$. If one considers the superluminal case $v > c$, then $\gamma^2 < 0$ and the question is whether (12), (14), and (1) – (5) have real solutions which satisfy the physical conditions. This is not obvious, because satisfying the conditions of the virial theorem only is necessary. The condition $v > c$ does not contradict the principles of special relativity, because it is not necessary to transmit signals inside the particle.

The following axisymmetric model is discussed. Both the electromagnetic field and the flow are stationary and axisymmetric. The electromagnetic field is poloidal, the current density and the flow are toroidal, while the mass and charge densities are axisymmetric scalars.

Let s, ϕ, z be cylindrical coordinates. The time-independent axisymmetric poloidal electromagnetic field is described by introducing a scalar potential $\Phi(s,z)$ and a flux function $\psi(s,z)$:

$$\vec{E} = -\nabla\Phi, \tag{16}$$

$$\vec{B} = \nabla\,\phi \times \nabla\,\psi. \tag{17}$$

This satisfies (1) and (2). Equation (3) is then Poisson's equation

$$\Delta\Phi = -\frac{q}{\epsilon_0}, \tag{18}$$

where the charge density $q(s,z)$ is an axisymmetric scalar and Δ is the Laplace operator in cylindrical coordinates:

$$\Delta = \frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} + \frac{\partial^2}{\partial z^2}. \tag{19}$$

Equation (4) reduces to

$$\Delta_\star \psi = \mu_0 s j_\phi(s,z), \tag{20}$$

where $j_\phi(s,z)$ is the toroidel component of the current density and

$$\Delta_\star = \frac{\partial^2}{\partial s^2} - \frac{1}{s}\frac{\partial}{\partial s} + \frac{\partial^2}{\partial z^2} \tag{21}$$

is the Stokes operator. If the contour lines ψ = const. in the poloidal plane ϕ = const. are closed curves, then the configuration is a torus. Let $s = s_1$ be the smallest and $s = s_2$ the largest value of s for such a curve. The ratio $A = (s_2 + s_1)/(s_2 - s_1)$ is called the aspect ratio of the torus. The current density vector field is

$$\vec{\jmath} = \frac{1}{\mu_0}(\Delta_\star\psi)\nabla\phi, \tag{22}$$

and from (5) the toroidal flow is related to the current density by

$$j_\phi = q\, v_\phi. \tag{23}$$

With the axisymmetric functions $\rho_0(s,z),\ v(s,z)$, the continuity equation (12) is satisfied. The term $(\vec{w}\cdot\nabla)\vec{w}$ with $\vec{w} = \gamma\vec{v}$ in the momentum (14) has only an s-component.

$$(\vec{w}\cdot\nabla)\vec{w} = -\frac{1}{s}\begin{pmatrix} w^2 \\ 0 \\ 0 \end{pmatrix} = -\frac{\gamma^2}{s}\begin{pmatrix} v^2 \\ 0 \\ 0 \end{pmatrix}, \tag{24}$$

where

$$\vec{v} = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}, \quad v(s,z) \tag{25}$$

is the axisymmetric toroidal flow. It is the sign of this factor γ^2 in (24) which affords hope of finding a solution for the superluminal case. With the Lorentz force

$$\vec{j} \times \vec{B} = -\frac{qv}{s} \nabla\psi, \tag{26}$$

the momentum equation (14) has the two poloidal components

$$\rho_0 \gamma^2 \frac{v^2}{s} = q(\Phi_s + \frac{v}{s}\psi_s), \tag{27}$$

$$0 = q(\Phi_z + \frac{v}{s}\psi_z), \tag{28}$$

where the subscripts s and z denote partial derivatives. So, finally we have the five equations (18), (20), (23), (27), and (28) for the six unknowns $\Phi, \psi, v, \rho_0, q, j_\phi$. Note, however, that we should consider an inside-outside free boundary value problem, where the toroidal interface has to be determined and outside the interface the PDEs

$$\Delta\Phi = \Delta_\star \psi = 0 \tag{29}$$

are valid. The aim is to construct a solution with finite energy. The total energy is

$$U = Mc^2 = U_M + U_E + U_B, \tag{30}$$

where

$$U_M = c^2 \int \rho_0 \gamma^2 \mathrm{d}^3\tau \tag{31}$$

is the mechanical energy, whose density is the 0,0-component of the kinetic energy-momentum tensor, and

$$U_E = \frac{\epsilon_0}{2} \int \vec{E}^2 \mathrm{d}^3\tau \tag{32}$$

$$U_B = \frac{1}{2\mu_0} \int \vec{B}^2 d^3\tau \tag{33}$$

are the electromagnetic energies. Here, these three energies should be finite, which means that point charge and line charge cannot be accepted, because both lead to unbounded forces and energies, while the energy of surface charge and volume charge is finite.

1.2. The Free Boundary Conditions

For deriving the free boundary conditions it is easiest to start from (15). Integrating (15) over the interior of the torus and applying the Gauss theorem to get the normal component of the stress, which must vanish, one obtains

$$\epsilon_0(E_i E_j n_j - \frac{1}{2}E^2 n_i) + \frac{1}{\mu_0}(B_i B_j n_j - \frac{1}{2}B^2 n_i) = 0, \tag{34}$$

where n_i is the outward pointing normal and the condition $v_i n_i = 0$ is used. In terms of Φ and ψ the normal and tangential components of (34) read

$$(\partial_n\Phi)^2 - (\partial_t\Phi)^2 - \frac{c^2}{s^2}[(\partial_n\psi)^2 - (\partial_t\psi)^2] = 0, \tag{35}$$

$$(\partial_n\Phi)\partial_t\Phi - \frac{c^2}{s^2}(\partial_n\psi)\partial_t\psi = 0, \tag{36}$$

with ∂_n the normal and ∂_t the tangential derivative. $s\partial_n\Phi = \pm c\partial_n\psi$, $s\partial_t\Phi = \pm c\partial_t\psi$ is a solution of (35) and (36). It is readily seen that it is the only real solution. This can be written in the form

$$s\nabla\Phi = \pm c\nabla\psi. \tag{37}$$

Integrating (37) yields

$$\psi(\Phi), \ \frac{\mathrm{d}\psi}{\mathrm{d}\Phi} = \pm\frac{s}{c} \tag{38}$$

or, if this is differentiated with respect to z, then

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\Phi^2}\Phi_z = 0, \tag{39}$$

from which it is concluded that

$$\Phi_z = \psi_z = 0. \tag{40}$$

The boundary condition (40) suggests trying to find solutions where all functions are independent of z everywhere in the interior.

1.3. The Interior Solution

Let us now consider reflexional symmetry with respect to the equatorial plane $z = 0$ and assume rigid rotation $v = \omega s$ with constant angular velocity ω. Then (28) is satisfied if both Φ and ψ are arbitrary functions of s. Using the abbreviation

$$\omega\psi(s) + \Phi(s) = f(s), \tag{41}$$

the solution of (27) is written in terms of f as

$$\rho_0 = \frac{qf'}{\gamma^2\omega^2 s}, \quad f' = \frac{\mathrm{d}f}{\mathrm{d}s}. \tag{42}$$

Eliminating q from (18), (20), and (23) yields

$$c^2\Delta_\star\psi + \omega s^2\Delta\Phi = 0. \tag{43}$$

Since the boundary conditions (40) are satisfied by assuming that all functions in the interior only depend on s in such a way that, because of

$$\Delta_\star\psi = s(\frac{1}{s}\psi')', \quad \Delta\Phi = \frac{1}{s}(s\Phi')', \tag{44}$$

(43) can be integrated once

$$\omega\psi' + \frac{v^2}{c^2}\Phi' = C_1 s \tag{45}$$

with an integration constant C_1 and a second time with a constant C_3 in the form of a quadrature:

$$\begin{aligned} &\omega\psi(s) + \frac{\omega^2}{c^2}s^2\Phi(s) \\ &-2\frac{\omega^2}{c^2}\int_{s_1}^{s}\Phi(u)u\mathrm{d}u = C_3 + \frac{1}{2}C_1 s^2. \end{aligned} \tag{46}$$

By eliminating $\omega\psi$ from (41) and (45) the mass density can be expressed by the electric field $E = -\Phi'$

$$\rho_0 = -\frac{\epsilon_0(sE)'}{\omega^2\gamma^2 s}(\frac{E}{s\gamma^2} - C_1). \tag{47}$$

Let the interface between the inside and outside be described by a non-negative function $Z(s)$ defined between the points s_1 and s_2. Then, if (47) is integrated over the interior volume, the z-integration can be performed by using the reflexional symmetry to give

$$W = 4\pi\int_{s_1}^{s_2} Z(s)\rho_0(s)s\mathrm{d}s. \tag{48}$$

W is a function which can be minimized with respect to the free parameters under the constraint that the mass

$$M = \int \rho_0 \mathrm{d}^3\tau > 0, \tag{49}$$

and the total charge $-e$ with $e > 0$ is fixed:

$$\begin{aligned} &-e = \int q\,\mathrm{d}^3\tau = 4\pi\int_{s_1}^{s_2} Z(s)q(s)s\mathrm{d}s, \\ &q = \epsilon_0(sE)'/s. \end{aligned} \tag{50}$$

The variation of W with respect to $E(s)$ yields the Euler equation

$$Z\{E[(\frac{1}{\gamma^4})' - \frac{2}{s\gamma^4}] - sC_1(\frac{1}{\gamma^2})'\} = 0. \tag{51}$$

Suppose that in (51) the expression in parentheses vanishes everywhere in the interior, implying that

$$\frac{E}{s\gamma^2} - C_1 = -C_1\frac{c^2}{c^2 + v^2}. \tag{52}$$

From (52) the derivative of sE

$$(sE)' = 4C_1\omega^2 c^6 \frac{s^3}{(v^4 - c^4)^2} \tag{53}$$

is found. So, this case leads to $\rho_0 \leq 0$, thus contradicting condition (49), which means that the minimum of W cannot be found with the aid of the Euler equation. A second possibility is to put the factor Z in (51) equal to zero and to consider the singular limit $Z(s) \to 0$. In this limit the internal region disappears.

1.4. The External Solution

In the outside region the axisymmetric vacuum fields are described by the scalar potential Φ and the flux function ψ which satisfy (29). The electric field $\vec{E}$ is produced by a volume charge density q in the interior V of the torus, and vanishes at infinity. For such a field Green's function $1/r$ is a fundamental solution of the Laplace equation, where in Cartesian coordinates the function r is the distance between the two points x, y, z and ξ, η, ζ:

$$r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2. \tag{54}$$

The potential Φ in terms of the volume charge density q is written as

$$4\pi\epsilon_0\Phi = \int_V \frac{q}{r}\mathrm{d}^3\tau, \tag{55}$$

which vanishes at infinity and for $r \neq 0$ satisfies the Laplace equation. In cylindrical coordinates s, ϕ, z and

t, χ, ζ with $\mathrm{d}^3\tau = t\mathrm{d}t\mathrm{d}\chi\mathrm{d}\zeta$ the χ-integral contains an elliptic integral

$$\int_0^{2\pi}[s^2+t^2-2st\cos(\phi-\chi)+(z-\zeta)^2]^{-\frac{1}{2}}\mathrm{d}\chi \\ = 4K(m)[(s+t)^2+(z-\zeta)^2]^{-\frac{1}{2}}, \tag{56}$$

where

$$K(m) = \int_0^{\frac{\pi}{2}}(1-m\sin^2\theta)^{-\frac{1}{2}}\mathrm{d}\theta \tag{57}$$

is the complete elliptic integral of the first kind with the parameter

$$m = \frac{4st}{(s+t)^2+(z-\zeta)^2}. \tag{58}$$

Thus, for Φ in terms of q we have the functional

$$\Phi(s,z) = \frac{1}{\pi\epsilon_0}\int_{s_1}^{s_2}\int_{-Z}^{Z}G(s,t,z-\zeta)q(t,\zeta)\mathrm{d}\zeta t\mathrm{d}t \tag{59}$$

with

$$G = [(s+t)^2+(z-\zeta)^2]^{-\frac{1}{2}}K(m). \tag{60}$$

The magnetic field $\vec{B}$ is produced by a volume current density $\vec{j}$ in V. It is related to the flow by

$$\vec{j} = q\vec{v}. \tag{61}$$

The axisymmetric Green's function H for the operator $\Delta_\star$ is derived with the aid of Biot-Savart's formula. It describes the magnetic field of a circular loop (see [5] and [6, p. 290]):

$$\psi(s,z) = -\frac{\mu_0\omega}{\pi}\int_{s_1}^{s_2}\int_{-Z}^{Z}H(s,t,z-\zeta)q(t,\zeta)\mathrm{d}\zeta t\mathrm{d}t, \tag{62}$$

$$H = \sqrt{\frac{st}{m}}[(1-\frac{m}{2})K(m)-E(m)]. \tag{63}$$

Here,

$$E(m) = \int_0^{\frac{\pi}{2}}(1-m\sin^2\theta)^{\frac{1}{2}}\mathrm{d}\theta \tag{64}$$

is the complete elliptic integral of the second kind with the parameter m in (58). The elliptical integral $E(m)$ should not be mixed up with the electric field $\vec{E}$ and its absolute value $E = |\vec{E}| = \sqrt{\vec{E}^2}$. For the electromagnetic field to be finite, continuity of Φ and ψ across the interface is assumed. Green's functions G and H behave at ∞ such that

$$E^2r^4 < \infty,\ B^2r^6 < \infty,\ r^2 = s^2+z^2, \tag{65}$$

which guarantees that at ∞ the electromagnetic energy density is integrable.

1.5. The Surface Charge Limit

In the limit $Z(s) \to 0$ the interior of the torus degenerates to a configuration like an accretion disk, where the charge is localized on the surface $S:\ s_1 \le s \le s_2,\ 0 \le \phi \le 2\pi,\ z = 0$, on which the volume charge density q tends to infinity in such a way that the surface charge density σ stays finite. In this limit the volume element $\mathrm{d}^3\tau$ is small in relation to the surface element d^2S. For the outside region the surface S is a "surface of discontinuity" on which the jump condition

$$\partial_n\Phi^+ - \partial_n\Phi^- = -\frac{\sigma}{\epsilon_0} \tag{66}$$

holds [6]. Let us define the relation between σ and q by

$$\sigma(t) = \lim_{Z\to 0}\int_{-Z(t)}^{Z(t)}q(t,\zeta)\mathrm{d}\zeta. \tag{67}$$

Then in the 2-dimensional integrals (59) and (62) the ζ-integration can be performed, yielding 1-dimensional integrals, and the ζ-dependence in (58)–(62) disappears

$$\Phi(s,z) = \frac{1}{\pi\epsilon_0}\int_{s_1}^{s_2}G(s,t,z)\sigma(t)t\mathrm{d}t, \tag{68}$$

$$\psi(s,z) = -\frac{\mu_0\omega}{\pi}\int_{s_1}^{s_2}H(s,t,z)\sigma(t)t\mathrm{d}t. \tag{69}$$

The function ψ satisfies the jump relation (see, for instance [7])

$$\mu_0\vec{j_S} = \vec{n}\times(\vec{B}_{\text{outside}}-\vec{B}_{\text{inside}}) = -(\psi_z^+-\psi_z^-)\nabla\phi. \tag{70}$$

If the integral representations (68) and (69) for Φ and ψ are inserted in (46), then for $\sigma(t)$ an integral equation of the form

$$\int_{s_1}^{s_2}J(s,t)\sigma(t)t\mathrm{d}t = C_4+C_2s^2,\ C_2 = \frac{C_1\pi}{2\mu_0\omega^2} \tag{71}$$

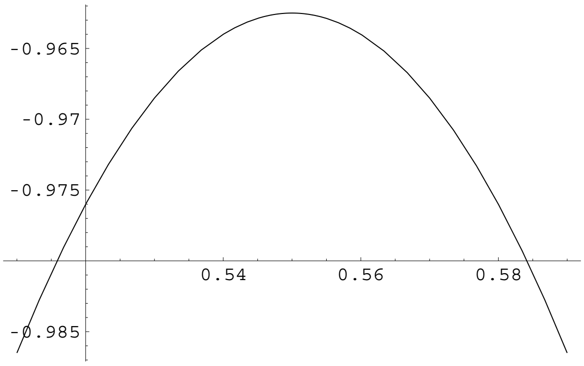

Fig. 1. Solution $\sigma_\star(s)$ of the integral equation.

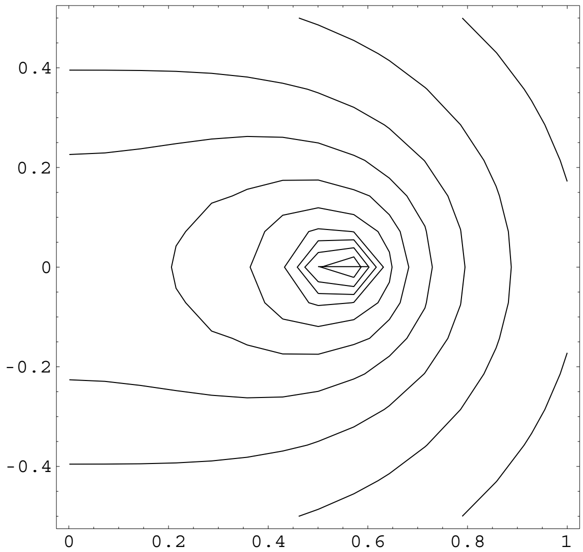

Fig. 2. Poloidal cut of the external equipotential surfaces Φ = const.

is obtained. The kernel J is

$$J(s,t) = -H(s,t,0) + s^2 G(s,t,0) - 2\int_{s_1}^{s} uG(u,t,0)\mathrm{d}u \tag{72}$$

1.6. Numerical Evaluation

The integral equation (71) is of the first kind. Solving this is not an easy problem, because it is ill-posed and numerically unstable [8, 9]. The kernel $J(s,t)$, which has a logarithmic singularity, defines a compact operator in $L_2[s_1, s_2]$ [10]. The Fredholm alternative states that if the homogeneous part of (71) has only the trivial solution, then the inhomogeneous equation has a square-integrable solution for every square-integrable r. h. s.

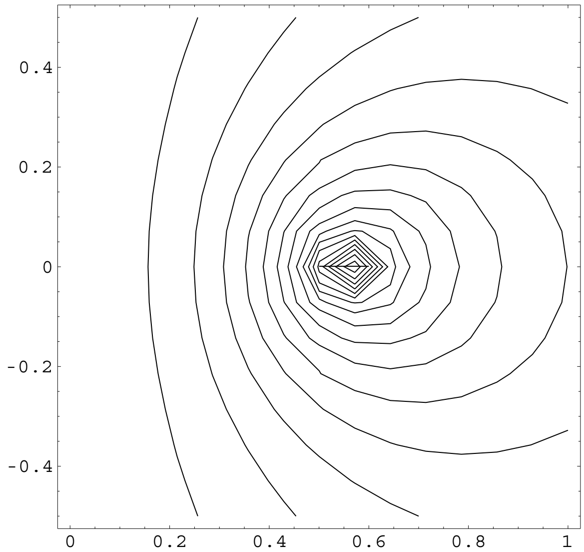

Fig. 3. External magnetic field lines ψ = const. in the poloidal plane.

In order to deal numerically with numbers of order unity, dimensionless quantities (with a hat) are introduced. With $R = e^2\mu_0/M$ the classical electron radius let

$$s = R\hat{s}, \quad \omega = \frac{c}{R}\hat{\omega}, \quad \Phi = \frac{e}{\pi R\epsilon_0}\hat{\Phi},$$
$$\psi = \frac{e}{\pi c\epsilon_0}\hat{\psi}, \quad q = \frac{e}{R^3}\hat{q},$$
$$\sigma = \frac{e}{R^2}\hat{\sigma}, \quad f = \frac{e}{\pi R\epsilon_0}\hat{f}, \quad H = R\hat{H},$$
$$C_1 = \frac{e}{\pi R^3\epsilon_0}\hat{C}_1, \quad C_2 = \frac{e}{R}\hat{C}_2,$$
$$\rho_0 = \frac{e^2\mu_0}{\pi R^4}\hat{\rho}_0, \quad U = \frac{e^2}{R\epsilon_0}\hat{U}, \quad \hat{\psi} = \hat{\omega}\bar{\psi}.$$

After dropping the hats one has

$$\bar{\psi}(s) = -\int_{s_1}^{s_2} H(s,t)\sigma(t)t\mathrm{d}t, \qquad \Phi(s) = \int_{s_1}^{s_2} G(s,t)\sigma(t)t\mathrm{d}t, \tag{73}$$

$$\bar{\psi}' + s^2\Phi' = 2C_2 s, \tag{74}$$

$$\omega^2\bar{\psi} + \Phi = f. \tag{75}$$

Equation (74) leads to the integral equation (71). Using Mathematica [11] this was numerically solved with

$C_2 = 1$, $C_4 = 0$ to give a function $\sigma_\star$, shown in Figure 1.

The constants C_2 and C_4 are determined by comparison with experimental data. The level lines of the external solutions Φ and $\bar{\psi}(s,z)$ are qualitatively shown in Figures 2 and 3.

The energies can be written in terms of $\sigma(s)$ by using (73) – (75) and the derivatives

$$G_s = -\frac{1}{2s}\left(\frac{K}{s+t} + \frac{E}{s-t}\right), \quad H_s = \frac{s}{2}\left(\frac{K}{s+t} - \frac{E}{s-t}\right).$$

For the total energy U and the mechanical energy U_{M} one obtains

$$U = 2\int_{s_1}^{s_2} f' \frac{\sigma \mathrm{d}s}{\gamma^2\omega^2} = \int_{s_1}^{s_2}\int_{s_1}^{s_2} \frac{1}{\omega^2 s^2}\left[\frac{K}{s+t}(\omega^4 s^4 - 1) - \frac{E}{s-t}(\omega^4 s^4 + 1)\right]\sigma(s)\sigma(t) st \mathrm{d}t \mathrm{d}s, \tag{76}$$

$$U_{\mathrm{M}} = 2\int_{s_1}^{s_2} f' \frac{\sigma \mathrm{d}s}{\omega^2} = -\int_{s_1}^{s_2}\int_{s_1}^{s_2} \frac{1}{\omega^2 s^2}\left[\frac{K}{s+t}(\omega^2 s^2 + 1) + \frac{E}{s-t}\right]\sigma(s)\sigma(t) st \mathrm{d}t \mathrm{d}s, \tag{77}$$

where the integrations over the poles are done with Cauchy's principle value. The electric energy U_{E} is

$$U_{\mathrm{E}} = \int_{s_1}^{s_2} \Phi\sigma s \mathrm{d}s = \int_{s_1}^{s_2}\int_{s_1}^{s_2} \frac{K}{s+t}\sigma(s)\sigma(t) st \mathrm{d}t \mathrm{d}s. \tag{78}$$

For the respective derivation of the magnetic energy

$$U_{\mathrm{B}} = -\omega^2 \int_{s_1}^{s_2} \bar{\psi}\sigma(s) s \mathrm{d}s = \frac{\omega^2}{2}\int_{s_1}^{s2}\int_{s_1}^{s_2}\left[\frac{K}{s+t}(s^2+t^2) - E(s+t)\right]\sigma(s)\sigma(t) st \mathrm{d}t \mathrm{d}s \tag{79}$$

the identity

$$\Delta_\star \psi = s^2 \mathrm{div}\frac{\nabla\psi}{s^2} \tag{80}$$

is used. Finally, with the aid of (73) the electromagnetic fields Φ and $\bar{\psi}$ are expressed by the surface charge density σ. So, all occurring quantities are written in terms of the solution $\sigma(t)$ of the integral equation (71).

1.7. Comparison with Experimental Data

The electron has a charge, mass, spin angular momentum, and electromagnetic field. The model parameters have to be related to these four quantities. In SI units we have the total charge

$$-e = \int q \mathrm{d}^3\tau = -1.602\cdot 10^{-19}\mathrm{As}, \; e > 0, \tag{81}$$

the total mass

$$M = \int \rho_0 \mathrm{d}^3\tau = 9.106\cdot 10^{-31}\mathrm{kg}, \; M > 0, \tag{82}$$

the spin angular momentum

$$L_z = \int (\vec{r} \times |\gamma|\rho_0\vec{v})\cdot\vec{e}_z \mathrm{d}^3\tau = \frac{1}{2}\hbar, \quad \hbar = 1.054\cdot 10^{-34}\mathrm{kg\,m^2/s}, \tag{83}$$

and the magnetic moment

$$M_z = \int (\vec{r} \times q\vec{v})\cdot\vec{e}_z \mathrm{d}^3\tau = -\nu\mu_{\mathrm{B}}, \quad \mu_{\mathrm{B}} = 9.2727\cdot 10^{-24}\mathrm{A\,m^2}. \tag{84}$$

Here, $\mu_B = e\hbar/(2M)$ is Bohr's magneton and the deviation from unity of the number $\nu = 1.00116$ describes the anomalous magnetic moment. Note that the vectors $\vec{M}$ and $\vec{L}$ have opposite direction, *i.e.* the gyromagnetic factor is negative [12, 13]. For comparison with the torus geometry (81) – (84) are written in dimensionless form as functionals of $\sigma(s)$:

$$\int_{s_1}^{s_2} \sigma(s) s \mathrm{d}s = -\frac{1}{2\pi}, \tag{85}$$

$$\int_{s_1}^{s_2} \sigma(s) s^3 \mathrm{d}s = -\frac{\nu}{16\pi^2\alpha|\omega|}, \tag{86}$$

$$\int_{s_1}^{s_2} \frac{f'}{\gamma^2}\sigma(s)\mathrm{d}s = \frac{1}{2}\omega^2, \tag{87}$$

$$\int_{s_1}^{s_2} \frac{f'}{|\gamma|}\sigma(s)s^2\mathrm{d}s = \frac{|\omega|}{16\pi\alpha}, \tag{88}$$

where $\alpha = e^2/(4\pi\hbar\epsilon_0 c) = 137.038^{-1}$ is the fine structure constant. The question is whether real parameters $s_1, s_2, C_2, C_4, \omega$ can be chosen in such a way that the four equations (85) – (88) are satisfied. If this could be answered in the affirmative, it would mean that the experimental data correspond to stationary points in parameter space. The electric dipole moment vanishes because of the reflexional symmetry in z. However, the quadrupole moment

$$D_{ij} = \int (3x_i x_j - x_k x_k \delta_{ij}) q \mathrm{d}^3\tau \tag{89}$$

is nonzero in the model and would be a candidate for experimentally testing the theory, which yields in dimensionless form

$$D_{zz} = -2\pi \int_{s_1}^{s_2} \sigma(s)s^3\mathrm{d}s = \frac{\nu}{8\pi\alpha\omega} \tag{90}$$

and $D_{xx} = D_{yy} = -\frac{1}{2}D_{zz}$.

At present the stability problem of the configuration is investigated. If it turned out that the experimental data correspond to a minimum of the energy, the theory could be considered as verified.

2. Conclusion

If the extended electron is classically described then for the case of stationary fields tending to zero at infinity the virial theorem states that the configuration cannot be in equilibrium if there are only electromagnetic forces present. Usually this problem is brushed aside by considering the configuration as a mathematical point (with radius zero). This, however, has the unpleasant consequence, mathematically as well as physically, that the self-energy diverges. If it is required that the energies be finite, then a point charge as well as a line charge cannot be accepted, because both lead to unbounded forces and energies, whereas the energies of a surface charge and a volume charge are finite. If one writes down the simplest relativistic forms of mass and energy-momentum balance for the case where there is only an electromagnetic force present, it is found that, if the system rotates with superluminal speed ($v > c$), there is a term with the factor γ^2 which has the opposite sign to the subluminal case $v < c$. It is this change of sign which makes it possible to bypass the virial theorem and find a stationary solution of the free-boundary value problem. The condition $v > c$ does not necessarily contradict the principles of relativity, because there is no need to transmit signals inside the particle. The question is: what is more fundamental inside a particle, the validity of electrodynamics or of relativistic mechanics? If one accepts the first alternative, then as a consequence a torus is needed because otherwise for $v > c$ there will be a singularity on the axis of rotation.

As a result of the theory, four dimensionless numbers are computed: 1) the ratio of the electromagnetic energy to the total energy, 2) the ratio of the electric energy to the magnetic energy, 3) the aspect ratio of the configuration, 4) the ratio $\omega s_1/c$ of the rotation speed to the speed of light. These will be presented in part II.

This model for the electron shows that quantization is not always necessary, and sometimes the description by stationary states of continuum mechanics may be simpler.

Acknowledgement

The author would like to thank his colleagues, D. Correa-Restrepo, D. Düchs, D. Pfirsch, G. Spies, and H. Wimmel of the former Theory Division of the Max-Planck-Institut für Plasmaphysik, for many fruitful discussions of the problem.

[1] H. Poincare, Comptes Rend. **140**, 1504 (1905).
[2] F. Rohrlich, Classical Charged Particles, Addison-Wesley Publishing Company, Reading, Mass. 1965.
[3] D. Lortz, Z. Naturforsch. **58a**, 491 (2003).
[4] D. Iwanenko and A. Sokolow, Klassische Feldtheorie, Akademie-Verlag, Berlin 1953.
[5] E. Martensen, Potentialtheorie, Teubner, Stuttgart 1968.
[6] W. R. Smythe, Static and Dynamic Electricity, Mc Graw-Hill Book Company, New York 1968, p. 304.
[7] G. Bateman, MHD Instabilities, The MIT Press, Cambridge/MA 1978, p. 76.
[8] H. Engl, Integralgleichungen, Springer-Verlag, Wien 1997.
[9] R. Kress, Linear Integral Equations, Springer-Verlag, Berlin 1989.

[10] H. Hochstadt, Integral Equations, John Wiley, New York 1973, p. 108.
[11] S. Wolfram, The Mathematica Book, 3rd ed., Wolfram Media/Cambridge University Press 1996.
[12] P. Dirac, The Principles of Quantum Mechanics, Clarendon Press, Oxford 1978, pp. 157, 266.
[13] R. Lerner and G. Trigg, Encyclopedia of Physics, VCH-Publishers, New York 1991, p. 467.