Steady Flow of an Electrically Conducting Incompressible Viscoelastic Fluid over a Heated Plate

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The transformation group theoretic approach is applied to the problem of the flow of an electrically conducting incompressible viscoelastic fluid near the forward stagnation point of a heated plate. The application of one-parameter transformation group reduces the number of independent variables, by one, and consequently the basic equations governing flow and heat transfer are reduced to a set of ordinary differential equations. These equations have been solved approximately subject to the relevant boundary conditions by employing the shooting numerical technique. The effect of the magnetic parameter $M$, the Prandtl number $Pr$ and the non-dimensional elastic parameter representing the non-Newtonian character of the fluid $k$ on velocity field, shear stress, temperature distribution and heat flux are carefully examined.

Key words: One-parameter Transformation Group; Viscoelastic Fluid; Non-Newtonian Fluid.

1. Introduction

Many attempts have been made to study the flow properties of non-Newtonian fluids. In 1969 Soundalgekar and Puri [1] have considered the interesting version of the problem of fluctuating flow of a non-Newtonian viscoelastic fluid under the condition of very small elastic parameter. The equations of motion for the steady state yield a third order non-linear differential equation, when the elasticity effect is taken into consideration, to be solved subject to two boundary conditions only. To overcome this difficulty, they [1] used a method which was developed by Beard and Walters [2] in 1964. They obtained the approximate solution valid for sufficiently small values of the elastic parameter by employing a perturbation procedure. The heat transfer aspect of this problem has been investigated by Massoudi and Ramezan [3] in 1992 and Garg [4] in 1994. In 1990 Garg and Rajagopal [5] and in 1994 Garg [4] have obtained solutions valid for all values of an elastic parameter by using an additional boundary condition at infinity, whereas Massoudi and Ramezan’s [3] work is confined to small values of elastic parameter.

In the present work we consider the flow of an electrically conducting incompressible viscoelastic fluid near the forward stagnation point of a solid plate. This type of problems has applications to engineering processes and polymer technology. The main purpose of this work is to study the effect of the magnetic parameter, the Prandtl number and the non-dimensional elastic parameter representing the non-Newtonian character of the fluid on velocity field, shear stress, temperature distribution and heat flux.

Similarity solutions are convenient methods to reduce systems of partial differential equations into systems of manageable ordinary differential equations. The mathematical technique used in the present analysis which leads to a similarity representation of the problem is the one-parameter group transformation. Group methods, as a class of methods which lead to a reduction of the number of independent variables, were first introduced by Birkhoff [6] in 1948, who made use of one-parameter transformation groups. Moran and Gaggioli [7, 8], in 1966 and 1969, presented a theory which has led to improvements over earlier similarity methods. Similarity analysis has been applied intensively by Gabbert [9] in 1967. For more additional dis-
cussions on group transformation, see Ames [10, 11], Bluman and Cole [12], Boisvert et al. [13], Gaggioli and Moran [14, 15]. Throughout the history of similar


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3.2. The Invariance Analysis

To transform the differential equation, transformations of the derivatives are obtained from \( G \) via chain-rule operations:

\[
S_i = \left( \frac{C^{*}}{C} \right) S_i, \quad S_{ij} = \left( \frac{C^{*}}{C C^i} \right) S_{ij},
\]

where \( S \) stands for \( \psi_0, \psi_1, U \) and \( T \) and \( S_i = \frac{\partial S}{\partial x^i} \).

Equations (2.6)–(2.8) are said to be invariantly transformed whenever

\[
\frac{\partial \psi_0}{\partial y} \frac{\partial^2 \psi_0}{\partial x \partial y} - \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_0}{\partial x^2} - \frac{\partial^3 \psi_0}{\partial x^3} + M \frac{\partial \psi_0}{\partial y} = H_1(a) \left[ \frac{\partial \psi_0}{\partial y} \frac{\partial^2 \psi_0}{\partial x \partial y} - \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_0}{\partial x^2} \right] + \frac{dU}{dx} \frac{\partial^3 \psi_0}{\partial y^3} + M \frac{\partial^2 \psi_0}{\partial y^2}
\]

(3.3)

\[
= H_2(a) \left[ \frac{\partial \psi_0}{\partial y} \frac{\partial^2 \psi_0}{\partial x \partial y} + \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_0}{\partial x^2} - \frac{\partial^3 \psi_0}{\partial x^3} \right] + M \frac{\partial \psi_0}{\partial y} \left( \frac{\partial^3 \psi_0}{\partial x \partial y^2} - \frac{\partial^2 \psi_0}{\partial x \partial y} \frac{\partial^2 \psi_0}{\partial x \partial y} - \frac{\partial^2 \psi_0}{\partial x \partial y} \frac{\partial^2 \psi_0}{\partial x^2 \partial y^2} \right)
\]

(3.4)

and

\[
= H_3(a) \left[ \frac{\partial \psi_0}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial \psi_0}{\partial x} \frac{\partial T}{\partial y} + k \left( \frac{\partial \psi_0}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial \psi_0}{\partial x} \frac{\partial T}{\partial y} \right) \right] - \frac{1}{Pr} \frac{\partial^2 T}{\partial y^2}
\]

(3.5)

for some functions \( H_1(a), H_2(a) \) and \( H_3(a) \) which depend only on the group parameter \( a \).

Substitution from (3.1) into (3.3)–(3.5) for the independent variables, the functions and their derivatives yields

\[
\left( \frac{C^{*}}{C} \right)^2 \frac{\partial \psi_0}{\partial y} \frac{\partial^2 \psi_0}{\partial x \partial y} - \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_0}{\partial x^2} - \frac{\partial^3 \psi_0}{\partial x^3} + M \frac{\partial \psi_0}{\partial y} = R_1(a)
\]

(3.6)

\[
\left( \frac{C^{*}}{C} \right)^2 \frac{\partial \psi_0}{\partial y} \frac{\partial^2 \psi_0}{\partial x \partial y} + \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_0}{\partial x^2} - \frac{\partial^3 \psi_0}{\partial x^3} \right] + M \frac{\partial \psi_0}{\partial y} \left( \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_0}{\partial x \partial y} - \frac{\partial^2 \psi_0}{\partial x \partial y} \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_0}{\partial x \partial y} \right)
\]

(3.7)

\[
\left( \frac{C^{*}}{C} \right)^2 \frac{\partial \psi_0}{\partial y} \frac{\partial^2 \psi_0}{\partial x \partial y} - \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_0}{\partial x^2} - \frac{\partial^3 \psi_0}{\partial x^3} \right] + M \frac{\partial \psi_0}{\partial y} \left( \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_0}{\partial x \partial y} - \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_0}{\partial x \partial y} \right)
\]

\[
\left( \frac{C^{*}}{C} \right)^2 \frac{\partial \psi_0}{\partial y} \frac{\partial^2 \psi_0}{\partial x \partial y} - \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_0}{\partial x^2} - \frac{\partial^3 \psi_0}{\partial x^3} \right] + M \frac{\partial \psi_0}{\partial y} \left( \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_0}{\partial x \partial y} - \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_0}{\partial x \partial y} \right)
\]
and
\[
\text{C}^{\psi_0} C^T = \left( \frac{\partial \psi_0}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial \psi_0}{\partial x} \frac{\partial T}{\partial y} \right) + k \frac{\partial \psi_1}{\partial y} \frac{\partial T}{\partial x} + k \frac{\partial \psi_1}{\partial x} \frac{\partial T}{\partial y} + \frac{1}{Pr (C^T)^2} \frac{\partial^2 T}{\partial y^2} + R_3(a)
\]
(3.8)

where
\[
R_1(a) = p_U \frac{C^U}{C^T} \frac{dU}{dx}, \quad R_2(a) = 0, \quad \text{and} \quad R_3(a) = 0.
\]
(3.9)

The invariance of (3.6) – (3.8) implies \( R_1(a) = R_2(a) = R_3(a) = 0 \). This is satisfied by putting
\[
P^U = 0,
\]
(3.12)

which yields
\[
C^T = 1, \quad C^{\psi_0} = C^T, \quad C^{\psi_1} = C^T
\]
(3.13)

Moreover, the boundary conditions (2.9) and (2.10) are also invariant in form, that implies
\[
P^T = P^T = 0 \quad \text{and} \quad C^T = 1.
\]
(3.14)

Finally, we get the one-parameter group \( G \) which transforms invariantly, the differential equations (3.6) – (3.8) and the boundary conditions (2.9) and (2.10). The group \( G \) is of the form
\[
G : \begin{cases}
\tilde{x} = C^x x + P^x,
\tilde{y} = y,
\psi_0 = C^{\psi_0} \psi_0 + P^{\psi_0},
\psi_1 = C^{\psi_1} \psi_1 + P^{\psi_1},
\tilde{U} = C^U U,
\tilde{T} = T.
\end{cases}
\]
(3.15)

3.3. The Complete Set of Absolute Invariants

Our aim is to make use of group methods to present the problem in the form of an ordinary differential equation (similarity representation) in a single independent variable (similarity variable). Then we have to proceed in analysis to obtain a complete set of absolute invariants. In addition to the absolute invariant of the independent variable, there are four absolute invariants of the dependent variables \( \psi_0, \psi_1, U \) and \( T \).

If \( \eta = \eta(x,y) \) is the absolute invariant of the independent variables, then
\[
g_j(x,y; \psi_0, \psi_1, U, T) = F_j[\eta(x,y)], \quad j = 1, 2, 3, 4,
\]
(3.16)

which are the four absolute invariants corresponding to \( \psi_0, \psi_1, U \) and \( T \). A function \( g = g(x,y; \psi_0, \psi_1, U, T) \) is an absolute invariant of a one-parameter group if it satisfies the following first-order linear differential equation
\[
\sum_{i=1}^{6} (\alpha_i S_i + \beta_i) \frac{\partial g}{\partial S_i} = 0,
\]
(3.17)

where \( S_i \) stands for \( x, y, \psi_0, \psi_1, U \) and \( T \), and
\[
\alpha_i = \frac{\partial C^{S_i}}{\partial a^0} (a^0), \quad \beta_i = \frac{\partial P^{S_i}}{\partial a^0} (a^0),
\]
(3.18)

where \( a^0 \) denotes the value of \( a \) which yields the identity element of the group.

From group (3.18) and using (3.21), we get:
\[
\alpha_2 = \beta_2 = \beta_5 = \alpha_6 = \beta_6 = 0.
\]
(3.19)

At first, we seek the absolute invariant of the independent variables. Owing to (3.20), \( \eta(x,y) \) is an absolute invariant if it satisfies the first-order linear partial differential equation
\[
(\alpha_1 x + \beta_1) \frac{\partial \eta}{\partial x} + (\alpha_2 y + \beta_2) \frac{\partial \eta}{\partial y} = 0,
\]
(3.20)
which reduces to
\[
\frac{\partial \psi}{\partial x} = 0. \quad (3.22)
\]

Equation (3.22) has a solution of the form
\[
\eta(x, y) = y. \quad (3.23)
\]

Similarly, analysis of the absolute invariants of the dependent variables \( \psi_0, \psi_1, U \) and \( T \) are
\[
\begin{align*}
\psi_0(x, y) &= I_0(x)F_0(\eta), \\
\psi_1(x, y) &= I_1(x)F_1(\eta), \\
U(x) &= I_2(x), \\
T(x, y) &= \theta(\eta). \\
\end{align*} \quad (3.24)
\]

### 3.4. The Reduction to Ordinary Differential Equation

As the general analysis proceeds, the established forms of the dependent and independent absolute invariant are used to obtain an ordinary differential equation. Generally, the absolute invariant \( \eta \) has the form given in (3.23).

Substituting from (3.24) into (2.6) yields
\[
\frac{d^3F_0}{d\eta^3} + \frac{dF_0}{d\eta} \left[ F_0 \frac{d^2F_0}{d\eta^2} - \left( \frac{dF_0}{d\eta} \right)^2 \right] - M \frac{dF_0}{d\eta} + \frac{I_2}{I_0} \frac{dF_2}{d\eta} = 0. \quad (3.25)
\]

For (3.25) to be reduced to an expression in a single independent variable \( \eta \), the coefficients in (3.25) should be constants or functions of \( \eta \). Thus,
\[
\frac{dF_0}{d\eta} = C_1, \quad (3.26)
\]
\[
\frac{I_2}{I_0} \frac{dF_2}{d\eta} = C_2. \quad (3.27)
\]

Assume \( C_1 = 1 \) and \( C_2 = U_0 \), where \( U_0 \) is an arbitrary constant, then \( I_0 = x \), and therefore \( I_2(x) = U_0x \) that actually obeys the power-law fluids. Hence, (3.25) reduces to
\[
\frac{d^3F_0}{d\eta^3} + F_0 \frac{d^2F_0}{d\eta^2} - \left( \frac{dF_0}{d\eta} \right)^2 - M \frac{dF_0}{d\eta} = -U_0^2. \quad (3.28)
\]

Substitute from the above results and from (3.24) into (2.7), we obtain
\[
\frac{d^4F_1}{d\eta^4} + F_0 \frac{d^3F_1}{d\eta^3} + \left( 1 + M \right) \frac{dF_1}{d\eta} + \frac{x}{I_1} \frac{dF_1}{dx} \left( \frac{d^2F_0}{d\eta^2} \frac{d^3F_1}{d\eta^3} - \left( \frac{dF_0}{d\eta} \right)^2 \right) = 0. \quad (3.29)
\]

Again, for (3.29) to be reduced to an expression in a single independent variable \( \eta \), the coefficients in (3.29) should be constants or functions of \( \eta \). Thus,
\[
\frac{x}{I_1} = C_3, \quad (3.30)
\]
\[
\frac{x}{I_1} \frac{dF_1}{dx} = C_4. \quad (3.31)
\]

Assuming \( C_3 = 1 \), then \( I_1 = x \), and therefore \( C_4 = 1 \), from which (3.29) takes the form
\[
\frac{d^4F_1}{d\eta^4} + F_0 \frac{d^3F_1}{d\eta^3} + \left( 2 \frac{dF_0}{d\eta} + M \right) \frac{dF_1}{d\eta} + \frac{d^2F_0}{d\eta^2} F_1 = 0. \quad (3.32)
\]

Finally, using (3.24), (2.8) will be converted to the following ordinary differential equation
\[
\frac{d^2\theta}{d\eta^2} + PrF \frac{d\theta}{d\eta} = 0, \quad (3.33)
\]

where \( F = F_0 + kF_1 \). Thus, under the similarity variable \( \eta \), (2.6) – (2.8) and their boundary conditions (2.9) and (2.10) will be transformed into the system of differential equations (3.28), (3.32) and (3.33) with the following appropriate corresponding conditions
\[
\begin{align*}
\eta = 0: & \quad F_0(\eta) = 0, \quad \frac{dF_0(\eta)}{d\eta} = 0, \\
F_1(\eta) = 0, & \quad \frac{dF_1(\eta)}{d\eta} = 0, \quad \theta(\eta) = T_0, \\
\eta \to \infty: & \quad \lim_{\eta \to \infty} \frac{dF_0(\eta)}{d\eta} = U_0, \\
\lim_{\eta \to \infty} \frac{dF_1(\eta)}{d\eta} = 0, & \quad \lim_{\eta \to \infty} \theta(\eta) = 0.
\end{align*} \quad (3.34)
\]
4. Numerical Results

For convenience let \( U_0 = T_0 = 1 \), the set of boundary value problem represented by (3.28), (3.32) and (3.33) under the appropriate conditions (3.34) and (3.35) has been solved numerically using the fourth-order Runge-Kutta shooting method. Having found \( F = F_0 + kF_1 \) from (3.28) and (3.32), the solution for (3.33) subject to its relevant conditions is obtained by a similar shooting method. We have an initial value problem from \( \eta_0 = 0 \) to \( \eta_\infty \), where \( \eta_\infty \) is a sufficiently large number.

Figure 1a shows the first approximation of stream function \( F_0 \) as a function of the similarity variable \( \eta \), for various values of magnetic parameter \( M \) and for fixed \( Pr = 0.7 \) and \( k = 0.2 \). It is noticed that the stream function \( F_0 \) decreases and comes close to each other as the magnetic parameter \( M \) increases.

Figure 1b shows the variation of first approximation of the velocity \( F'_0 \) with \( \eta \) for various values of magnetic parameter \( M \) and for fixed \( Pr = 0.7 \) and \( k = 0.2 \). It is clear that the velocity of the fluid decreases with increasing the magnetic parameter. In addition, this figure shows that the smaller the value of \( M \) the faster it reaches the maximum value of \( F'_0 \).

Figure 1c shows the variation of first approximation of the shear stress \( F''_0 \) with \( \eta \) for various values of magnetic parameter \( M \) and for fixed \( Pr = 0.7 \) and \( k = 0.2 \). It is obvious that the shear stress changes depending on the magnetic parameter and the distance; the shear stress decreases with increasing the magnetic parame-
For a small value of $M$, the shear stress starts with a high value then decreases with increasing the distance. For a high value of $M$, the shear stress starts with a lower value and decreases with the distance.

Figure 2b shows the variation of second approximation of the velocity $F'_1$ with $\eta$ for various values of magnetic parameter $M$ and for fixed $Pr = 0.7$ and $k = 0.2$. It is clear that $F'_1$ overshoots for small values of $\eta$ and then decreases to zero for large $\eta$. In addition, this figure shows that the smaller the value of $M$ the faster it reaches the maximum value of $F'_1$.

Figure 2c shows the variation of second approximation of the shear stress $F''_1$ with $\eta$ for various values of magnetic parameter $M$ and for fixed $Pr = 0.7$ and $k = 0.2$. It is obvious that $F''_1$ decreases with increasing the magnetic parameter for small values of $\eta$ and then becomes negative in a certain region and increases to zero for large $\eta$.

The variation of the temperature $\theta$ with $\eta$ is illustrated in Figure 3a. The results are obtained for $M = 0$, 0.5, 1, 2 and corresponding to $Pr = 0.7$ and $k = 0.2$. The figure shows the rapid decrease of the temperature distribution at $M = 0$. Also, the temperature increases with increasing the magnetic parameter $M$.
Figure 3b shows the variation of the heat flux $-\theta'$ with $\eta$. It is clear that the heat flux starts with a higher value for the lower values of the magnetic parameter $M$ and then decreases.

The effect of $Pr$ on the temperature and heat flux is illustrated in Figs. 4a and 4b. The results are obtained for $Pr = 0.7, 2, 6$ and 10. For the temperature profile, Fig. 4a indicates the occurrence of the rapid decrease in $\theta$. This becomes more evident for larger values of $Pr$. Also, Fig. 4b shows the rapid decrease in the heat flux for increasing values of $Pr$.

Figure 5a shows the variation of the temperature $\theta$ with $\eta$ and Fig. 5b shows the variation of the heat flux $-\theta'$ with $\eta$ for various values of non-dimensional elastic parameter $k$. It is apparent from Fig. 5a that the temperature profiles slightly decrease with an increase in the elasticity of the fluid, but in Fig. 5b the heat flux distribution varies (higher-lower) at different values of the elasticity of the fluid. Again from Figs. 5a and 5b, we arrive to the conclusion that the effect of non-dimensional elastic parameter $k$ is still small for the increase in the Prandtl number.

5. Conclusion

The group method confirmed that it is a powerful tool for solving the problems of magneto-elastic-viscous flow near the forward stagnation point of a solid plate with heat transfer and obtaining the velocity profiles, shear stress and heat flux for various values of the magnetic parameter. Numerical results of the transformed boundary layer equations have been obtained by using the Runge-Kutta shooting method. Referring to the numerical results and the figures it is observed that:

(i) We observe from Figs. 1 and 2 that the main effect of increasing the magnetic parameter $M$ on the two dimensional magneto-elastic-viscous flow is to decrease the velocity field and shear stress in the direction of the solid plate.

(ii) From Fig. 5, we arrive at the conclusion that the thermal boundary layer thickness becomes small for the increase in the Prandtl number $Pr$.

(iii) Our perturbation analysis is valid only for small values of elastic parameter $k$.

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