Non-polynomial Third Order Equations which Pass the Painlevé Test

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The singular point analysis of third-order ordinary differential equations in the non-polynomial class is presented. Some new third order ordinary differential equations which pass the Painlevé test, as well as the known ones are found.

Key words: Painlevé Equations, Painlevé Test.

1. Introduction

Painlevé and his school [1 – 3] studied a certain class of second order ordinary differential equations (ODE’s) and found fifty canonical equations whose solutions have no movable critical points. This property is known as the Painlevé property. Distinguished among these fifty equations are six Painlevé equations, PI-PVI, which are regarded as nonlinear special functions.

The third order Painlevé type equations

\[ y''' = F(z, y, y', y''), \]  

(1.1)

where \( F \) is polynomial in \( y \) and its derivatives, were considered in [4 – 7]. Some fourth and higher order polynomial-type equations with the Painlevé property were investigated in [5 – 10].

The third order equation (1.1), such that \( F \) is analytic in \( z \) and rational in its other arguments, was considered in [11, 12]. [12] starts with a simplified equation, i.e. an equation which contains terms with leading order \( \alpha = -1 \) as \( z \to z_0 \) only:

\[ y''' = \left(1 - \frac{1}{y'}\right) \frac{(y'' - 2yy'^2)}{y'' - y'^2} + c_1 \frac{y'''}{y}, \]  

(1.2)

where \( \alpha = -1 \) as \( z \to z_0 \) only:

\[ y''' = \left(1 - \frac{1}{y'}\right) \frac{(y'' - 2yy'^2)}{y'' - y'^2} + c_1 \frac{y'''}{y}, \]  

(1.2)

where \( a_i = \) constant, \( i = 1, 2, 3, 4 \), \( \nu \in \mathbb{Z} - \{-1, 0\} \), \( c_j = \) constant, \( j = 1, 2, \) \( c_1^2 + c_2^2 \neq 0 \), and investigates the values of \( a_i \), and \( c_j \) such that the equation is of Painlevé type.

We consider the third order differential equation

\[ y''' = c_1 \frac{y'y''}{y} + c_2 \frac{(y')^3}{y^2} + F(y, y', y''; z), \]  

(1.3)

where \( c_1 \) and \( c_2 \) are constants, such that \( c_1^2 + c_2^2 \neq 0 \). \( F \) may contain the leading terms, but all the terms of \( F \) are of order \( \varepsilon^{-2} \) or greater if we let \( z = z_0 + \varepsilon t \), where \( \varepsilon \) is a small parameter, \( t \) is the new independent variable and the coefficients of \( F \) are locally analytic functions of \( z \). The equation of type (1.3) can be obtained by differentiating the leading terms of the third Painlevé equation and adding the terms of order \( -4 \) or greater as \( z \to z_0 \) with the analytic coefficients in \( z \) such that: i.) \( y = 0, \infty \) are the only singular values of the equation in \( y \), ii.) The additional terms are of order \( \varepsilon^{-3} \) or greater, if one lets \( z = z_0 + \varepsilon t \).

If we let \( z = z_0 + \varepsilon t \) and take the limit as \( \varepsilon \to 0 \), (1.3) yields the “reduced” equation

\[ \dddot{y} = c_1 \frac{\ddot{y}y''}{y} + c_2 \frac{y''^3}{y'^2}, \]  

(1.4)

where \( \dot{\cdot} = \frac{d}{dt} \). Substituting \( y \equiv y_0(t-t_0)^\alpha \) into (1.4) gives

\[ (c_1 + c_2 - 1)\alpha^2 - (c_1 - 3)\alpha - 2 = 0. \]  

(1.5)

Let \( c_1 + c_2 - 1 \neq 0 \) and the roots of (1.5) be \( \alpha_1 = n \) and \( \alpha_2 = m \) such that \( n, m \in \mathbb{Z} - \{0\} \), then

\[ (1 - m - n)c_1 - (n + m)c_2 + m + n - 3 = 0, \]  

(1.6)

\[ (n - m)(c_1 + c_2 - 1)^2 - c_1(c_1 + 2) - 8c_2 - 1 = 0. \]
If \( n + m - 1 \neq 0 \), then
\[
(c_2 + 2)[2(1 - m - n + mn) + mn c_2] = 0. \tag{1.7}
\]
It should be noted that if \( c_2 = -2 \), then \( c_1 = 3 \) and \( c_1 + c_2 - 1 = 0 \). So we have
\[
(c_1, c_2) = \left( \frac{1}{mn}(3mn - 2n - 2m), \frac{2}{mn}(m + n - mn - 1) \right) \tag{1.8}
\]
when \( n + m - 1 \neq 0 \), \( c_1 \neq 3 \) and \( c_1 + c_2 - 1 \neq 0 \).
Substituting
\[
y = y_0(t - t_0)^\alpha + \beta (t - t_0)^{r + \alpha}
\]
into (1.4), we obtain the equations for the Fuchs indices in the form
\[
r(r + 1)[mr + 2(n - m)] = 0, \quad r(r + 1)[nr - 2(n - m)] = 0 \tag{1.10}
\]
for \( \alpha = n \) and \( \alpha = m \), respectively. So, the Fuchs indices are,
\[
(r_0, r_1, r_2) = \left( -1, 0, 2 - \frac{2n}{m} \right),
\]
\[
(r_0, r_1, r_2) = \left( -1, 0, 2 - \frac{2m}{n} \right). \tag{1.11}
\]
for \( \alpha = n \) and \( \alpha = m \) respectively. In order to have distinct indices, if \( p = 2n/m \) or \( q = 2m/n \) than \( p, q \in \mathbb{Z} \) and satisfy the Diophantine equation \( pq = 4 \). By solving the Diophantine equation for \( p, q \) and using the symmetry of (1.8) with respect to \( n \) and \( m \), one gets the following 3 cases for \( (c_1, c_2) \):

1. \( (c_1, c_2) = \left( 3, -2 + \frac{2}{n} \right) \),
2. \( (c_1, c_2) = \left( 3 - \frac{1}{n}, -2 + \frac{1}{n} + \frac{1}{n^2} \right) \tag{1.12.a,b,c} \),
3. \( (c_1, c_2) = \left( 3 - \frac{2}{n}, -2 + \frac{3}{n} - \frac{1}{n^2} \right) \).

If \( n + m - 1 = 0 \), (1.6) and \( c_1 + c_2 - 1 \neq 0 \) imply that \( c_2 = -2 \) and \( c_1 \neq 3 \), respectively. Then
\[
(c_1, c_2) = \left( \frac{3n^2 - 3n + 2}{n(n - 1)}, -2 \right), \tag{1.13}
\]
\( n \neq 0, 1 \), and \( c_1 \neq 3 \).

Similarly, substituting (1.9) into (1.4) with the values of \( (c_1, c_2) \) given in (1.13) gives the equations for the Fuchs indices in the form
\[
r(r + 1)[r(n - 1) + 2(1 - 2n)] = 0, \quad r(r + 1)[nr - 2(1 - 2n)] = 0 \tag{1.14}
\]
for \( \alpha = n \) and \( \alpha = m = 1 - n \), respectively. In order to have distinct Fuchs indices for both branches \( \alpha = n \) and \( \alpha = m \), \( n \) must take the values \(-1, 2 \). Therefore, when \( n + m - 1 = 0 \) and \( c_1 + c_2 - 1 \neq 0 \) we have \( (c_1, c_2) = (4, -2) \), which can be obtained from (1.12b) for \( n = -1 \).

In the case of the single branch, i.e. \( c_1 + c_2 - 1 = 0 \), let \( \alpha = n \in \mathbb{Z} - \{0\} \), the Fuchs indices are \( r = -1, 0, 2 \), and the coefficients \( (c_1, c_2) \) are
\[
4. \quad (c_1, c_2) = \left( 3 - \frac{2}{n}, -2 + \frac{2}{n} \right). \tag{1.15}
\]
If \( c_1 + c_2 - 1 = 0 \) and \( c_1 = 3 \), then \( c_2 = -2 \). So, as the fifth case we have
\[
5. \quad (c_1, c_2) = (3, -2). \tag{1.16}
\]
Thus, we have five cases, (1.12), (1.15) and (1.16), and all the corresponding equations pass the Painlevé test. Moreover, if one lets \( y = u^n \) in (1.4) with the coefficients \( (c_1, c_2) \) given by (1.12) and (1.15), and integrates the resulting equation for \( u \) once, then \( u \) satisfies a linear equation or is solvable by means of elliptic functions. For \( (c_1, c_2) \) given by (1.16), (1.4) yields \( u = 0 \) if we let \( u = y/y \) and integrate the resulting equation twice. Therefore all five equations have the Painlevé property.

2. Leading Order \( \alpha = -1 \)

Equation (1.4) contains the leading terms for any \( \alpha \in \mathbb{Z} - \{0\} \) as \( z \rightarrow \infty \). In this section we consider the case \( \alpha = -1 \). By adding the terms of order \(-4\) or greater as \( z \rightarrow \infty \), we obtain the equation
\[
y'' = c_1 y'' + c_2 \left( y' \right)^3 + a_1 y'' + a_2 (y')^2 + a_3 y y' + a_4 y^3 + F_j(y, y', y'', z), \tag{2.1}
\]
where \( a_i, i = 1, \ldots, 4 \) are constants and \( F_j, j = 1, 2; \)
\[
F_1 = A_1 y'' + A_2 \left( y' \right)^2 \frac{y}{y} + A_3 y' y + A_4 y^3 + A_5 \frac{y''}{y} + A_6 y^2 + A_7 y + A_8 y^2 + A_9 y + A_{10} + A_{11},
\]
\[
F_2 = A_1 y'' + A_2 \left( y' \right)^2 \frac{y}{y} + A_3 y' y + A_4 y^3 + A_5 \frac{y''}{y} + A_6 y^2 + A_7 y + A_8 y^2 + A_9 y + A_{10} + A_{11} y
\]
if \( c_2 = 0 \) and \( c_2 \neq 0 \), respectively, and where \( A_k(z) \) are locally analytic functions of \( z \). (2.1) contains all the leading terms for \( \alpha = -1 \), if we do not take into account \( F_j \).

Suppose that (1.12), (1.15) and (1.16) hold and substitute [13]

\[
y \equiv y_0(z-z_0)^{-1} + \beta(z-z_0)^{r-1} \tag{2.3}
\]

into (2.1) without \( F_1 \). Then we obtain the following equations for the Fuchs indices (resonances) \( r \) and \( y_0 \)

\[
Q(r) = (r+1)(r^2 - (a_1y_0 + 7 - c_1)r + 3(6 - 2c_1 - c_2) + 2(a_1 + a_2)y_0 - a_3y_0^2) = 0, \tag{2.4.a,b}
\]

\[
a_4y_0^3 - a_3y_0^2 + (2a_1 + a_2)y_0 + 6 - 2c_1 - c_2 = 0,
\]

respectively. Equation (2.4.b) implies that, in general, there are three branches if \( a_4 \neq 0 \). Now we determine \( y_0j \), \( j = 1, 2, 3 \), and \( \alpha_i \), \( i = 1, 2, 3, 4 \), for each case of \( (c_1, c_2) \) such that at least one branch is the principal branch, i.e. all the resonances are positive and distinct integers (except \( r_0 = -1 \)). \( A_k \) can be determined by using the transformation

\[
y = \mu(z)\tilde{y}(x), \quad x = \rho(z), \tag{2.5}
\]

which preserves the Painlevé property, where \( \mu \) and \( \rho \) are locally analytic functions of \( z \) and the compatibility conditions at the Fuchs indices \( r_{ji} \) and the compatibility conditions corresponding to parametric zeros; that is, the compatibility conditions at the Fuchs indices \( \bar{r}_{ji} \) of the equations obtained by the transformation \( y = 1/u \).

According to the number of branches, the following cases should be considered separately.

**Case I.** \( a_1 = a_2 = 0 \): In this case there is one branch. If \( r_0 = -1 \) and \( (r_1, r_2) \) are resonances, then (2.4.b) implies that

\[
-(2a_1 + a_2)y_0 = r_1r_2 = 6 - 2c_1 - c_2, \quad r_1 + r_2 = a_1y_0 + 7 - c_1. \tag{2.6}
\]

In order to have a principal branch,

\[
6 - 2c_1 - c_2 = k, \quad k \in \mathbb{Z}_+. \tag{2.7}
\]

When \( (c_1, c_2) = (3, -2 + \frac{2}{5}) \), (2.7) implies that \( n = \pm 1 \). Then \( y_0 \neq 0 \) and arbitrary, the Fuchs indices are \( (r_1, r_2) = (0, 4) \) and the simplified equation is [12]

\[
y''' = 3\frac{y''y}{y}.
\]

Integrating (2.8) once yields \( y'' = k_1y^3 \), where \( k_1 \) is an integration constant.

In this case, the canonical form of the equation is

\[
y''' = 3\frac{y''y}{y} + A_1y'' + A_2(\frac{y'}{y})^2 + A_3yy' + A_4y^3 + A_5\frac{y''}{y} + A_6y^2 + A_7y + A_8 + A_9 + A_{10} + A_{11}\frac{1}{y}. \tag{2.9}
\]

\( A_3 = 0 \), otherwise \( \alpha = -2 \) is of leading order. The transformation (2.5) allows one to take \( A_1 = A_2 = 0 \). If we substitute

\[
y = (z - z_0)^{-1} + \sum_{\ell = 0}^{\infty} y_\ell(z - z_0)^\ell \tag{2.10}
\]

in (2.9), then the compatibility condition at \( r_2 = 4 \) gives that \( A_4 = A_7 \neq 0 \) and

\[
A_0^2 + A_{10} - A_4 = 0, \quad A_6^2 - 2A_6 = 0. \tag{2.11}
\]

If we let \( y = 1/u \), then (2.9) yields

\[
u'' = 3u' u'' + A_5 \left[u^2 u'' - 2u(u')^2\right] + A_6 uu''
+ A_8u^2 u' - A_9u^2 - A_{10}u^3 - A_{11}u^4. \tag{2.12}
\]

\( \alpha = -1 \) is the possible leading order of \( u \) as \( z \to z_0 \), if \( A_5 = A_{11} = 0 \) and the Fuchs indices are \( (\tilde{r}_1, \tilde{r}_2) = (0, 4) \). The compatibility condition at \( \tilde{r}_2 = 4 \) together with (2.11) gives \( A_8 = k_1 = \text{constant}, A_{10} = 0, A'_6 = A''_6 = 0 \) and

\[
k_1(A'_6 + 2A_6) = 0. \tag{2.13}
\]

If \( k_1 = 0 \), then the canonical form of the equation is

\[
yyy = 3y'y'' + (kz + k_3)yy' + k_4y^2. \tag{2.14}
\]

If one lets \( y = e^v \) and \( v' = w \), then (2.14) yields the second Painlevé equation. If \( k_1 \neq 0 \), then we have

\[
yyy = 3y'y'' - (2kz_k - k_3)yy' + k_1y' + k_2y^2, \tag{2.15}
\]

where \( k_i = \text{constant for} i = 2, 3 \). Integrating (2.15) once yields

\[
y'' = k_4y^3 + \frac{1}{2}(2k_2z_k - k_3)y - \frac{k_1}{3}, \quad k_4 = \text{constant}. \tag{2.16}
\]
(2.16) is of Painlevé type [14]. When \((c_1, c_2) = (3 - 1/n, -2 + 1/n + 1/n^2)\), (2.7) implies that \(n = \pm 1\). For \(n = -1\), \(y_0 = 0\); for \(n = 1\), \(y_0 = 0\); and \(r_1 = 0\), \(r_2 = 1\).

\[
y''' = 4\frac{y'y''}{y} - 2\left(\frac{y'}{y}\right)^3. \tag{2.17}
\]

Integration of (2.17) once yields

\[
y'' = \frac{1}{2} \left(\frac{y'}{y}\right)^2 + k_1y^3, \quad k_1 = \text{constant}, \tag{2.18}
\]

which is solvable by means of elliptic functions.

After adding the non-dominant terms \(F_2\) given by (2.2.b), the leading order is \(\alpha = -1\) if \(A_1 = 0\). The compatibility condition at \(r_2 = 3\) implies that \(A_5 = A_8 = 0\). On the other hand, if \(A_5 = 0\), then the leading order of \(u = y/z\) as \(z \to z_0\) is \(\tilde{\alpha} = -1\). The following two cases may be considered separately:

If \(A_1 = 0\), then \((2.22)\) was considered in [12].

When \(A_2 = A_15 = 0\) and \(A_12(z) \neq 0\), then \(u = (1/y) \sim (z - z_0)^{-1}\) as \(z \to z_0\) \((\tilde{\alpha} = -1)\). The Fuchs indices are \((\tilde{r}_1, \tilde{r}_2) = (3, 4)\) and the canonical form of the equation is

\[
y^2y''' = 6yy'y'' - 6(y')^3 + 4(z^2 + k_1)y^2y' \nonumber
\]

\[
+ 12yz^2y' - 4z^3y + 6y' + 4y^2, \tag{2.23}
\]

where \(k_1\) is a constant. If \(A_9 = A_12 = A_14 = A_15 = 0\) and \(A_{10}(z) \neq 0\), then \(\tilde{\alpha} = -2\). \((\tilde{r}_1, \tilde{r}_2) = (4, 6)\) [6], and the canonical form of the equation is

\[
y^2y''' = 6yy'y'' - 6(y')^3 + \frac{1}{z} \left[2y''y - 2(y')^2\right] \nonumber
\]

\[
+ \left(\frac{1}{z} + 6z^3\right)y^4 + 12yz^2y - 12z^3y + 6\frac{y^4}{z^2}. \tag{2.24}
\]

where \(k_1\) and \(k_2\) are constants. If \(A_9 = A_{10} = A_{12} = A_{13} = A_{14} = A_{15} = 0\), then \(u\) satisfies a linear equation and the canonical form of (2.24) is

\[
y^2y''' = 6yy'y'' - 6(y')^3 + A_1y^2y''' - 2y(y')^2 \nonumber
\]

\[
+ A_7y^3y' + A_8y^4 + A_{11}y^3, \tag{2.25}
\]

where \(A_1, A_7, A_8, A_{11}\) are arbitrary locally analytic functions of \(z\).

When \((c_1, c_2) = (3 - 2/n, -2 + 2/n)\), (2.7) implies that \(n = \pm 1, \pm 2\). For \(n = -1\), let \(r_1 = 0\), then \(y_0 \neq 0\); \(r_2 = 2\) and the simplified equation is

\[
y''' = 3\frac{y'y''}{y} - 4\left(\frac{y'}{y}\right)^3. \tag{2.26}
\]

Integration of (2.26) once yields

\[
y'' = \frac{(y')^2}{y} + k_1y^3, \quad k_1 = \text{constant}, \tag{2.27}
\]

which is solvable by means of elliptic functions.
If we add the non-dominant terms $F_2$ given in (2.2.b) to (2.26) then we should set $A_3 = 0$, in order to have the leading order $\alpha = -1$. The transformation (2.5) and the compatibility condition at $r_2 = 2$ imply that $A_5 = A_6 = 0$ and $A_1 = A_8 = 0$, respectively. On the other hand, if $A_9 = A_{15} = 0$ and $A_{12} \neq 0$, then $\tilde{\alpha} = -1$, $A_{12,0} \rho^2 = 4$ and $(\tilde{r}_j, \tilde{r}_k) = (2, 4)$. The compatibility conditions at $\tilde{r}_j$ imply that all the coefficients $A_k$ are zero except $A_{10} = k_1$ and $A_{12} = k_2$, $k_1 = \text{constant}, i = 1, 2$. Therefore, the canonical form of the equation is

$$y^2y'' = 5yy'' - 4(y')^3 + k_1yy' + k_2y'. \quad (2.28)$$

For $n = 1$, (2.6) implies that $r_{1r_2} = 4$. Then the Fuchs indices and the simplified equation are

$$y_0 = -\frac{1}{a_1}: \quad (r_1, r_2) = (1, 4), \quad y'' = y'(\frac{y'}{y})^2 + a_1 [y'' + 2(y')^2]. \quad (2.29.a,b)$$

(2.29) was also considered in [12]. If one replaces $y$ by $\lambda y$ such that $a_1 \lambda = -1$ and lets $y = 1/u$, (2.29) yields

$$u^2u'' = 5uu'' - 4(u')^3 - uu'' + 4(u')^2. \quad (2.30)$$

(2.30) does not pass the Painlevé test. Hence (2.30), and consequently (2.29) is not of Painlevé type.

For $n = 2$, $y_0$, the Fuchs indices and the simplified equation are

$$y_0 = -\frac{1}{a_1}: \quad (r_1, r_2) = (1, 3), \quad y'' = 2(\frac{y'}{y})^3 + (\frac{y'}{y})^2 + a_1 [y'' + (y')^2], \quad (2.31.a,b)$$

respectively. (2.31) was considered in [12], and its first integral is

$$y'' = (\frac{y'}{y})^2 + a_1yy' + k_1, \quad k_1 = \text{constant}. \quad (2.32)$$

(2.32) is of Painlevé type [3, 14].

If we add the non-dominant terms to (2.31), the leading order $\alpha$ depending on of $u$ as $z \rightarrow z_0$, we have the following canonical form of the equations: If $\tilde{\alpha} = -1$ then $(\tilde{r}_j, \tilde{r}_k) = (1, 2), j = 1, 2$. The transformation (2.5), the compatibility conditions at $\tilde{r}_j$, $i, j = 1, 2$, and the compatibility conditions at $(r_1, r_2) = (1, 3)$ are enough to determine all the coefficients $A_k$ in terms of $A_1$. Then, one gets the following canonical form of the equation

$$y^2y'' = 2yy'' - (y')^3 - y'y'' - y(y')^2$$

$$+ A_1 [y^2y'' - y(y')^2 + y'y'] + A_1'y^2y' \quad (2.33)$$

$$+ (A_1'' - A_1A_1')y^3 + A_{12}(y' + y^2) - A_1A_{12}y,$$

where $A_{12}' = 2A_1A_{12}$. If $\alpha = -2$, then $A_5 = A_6 = A_10 = A_{12} = A_{15} = 0$, and $(\tilde{r}_1, \tilde{r}_2) = (0, 2)$. The compatibility condition at $\tilde{r}_2 = 2$ gives that $A_8 = A_{13} = 0$ and $A_3 = A_1', A_{11} = A_7 - A_1A_1'$. Then, the canonical form of the equation is

$$y^2y'' = 2yy'y'' - (y')^3 - y'y'' - y(y')^2$$

$$+ A_1 [y^2y'' - y(y')^2 + y'y'] + A_1'y^2y'$$

$$+ (A_1'' - A_1A_1')y^3, \quad (2.34)$$

where $A_1$ is a locally analytic function of $z$.

For $n = -2$, since $r_{1r_2} = 1$, $r = \pm 1$ are the double Fuchs indices.

When $(c_1, c_2) = (3, -2), y_0$, the Fuchs indices and the simplified equation are

$$y_0 = -\frac{1}{a_1}: \quad (r_1, r_2) = (1, 2), \quad y'' = 3(\frac{y'}{y})^3 + a_1yy'' \quad (2.35)$$

(2.35) was also considered in [12].

If one adds the non-dominant terms, then $\tilde{\alpha} = -1$ when $A_6 = -2A_5, A_9 = A_{12} = A_{15} = 0$ and $A_3(z_0)u_0 = -1, (\tilde{r}_1, \tilde{r}_2) = (1, 2)$. Therefore, the canonical form of the equation is

$$y^2y'' = 3yy'' - 2(y')^3 - y'y'$$

$$+ A_1 [y^2y'' - 2y(y')^2 - y'y' - y^5]$$

$$+ A_5 [y'' - 2(y')^2] + A_7(y^2y' + y^4)$$

$$+ (2A_5' - 3A_6A_5)yy'' + A_1'y^3$$

$$- (A_5'' - A_1A_5' - A_5A_7)y^2 - A_7A_7y, \quad (2.36)$$

where $A_1, A_5, A_7$ and $A_{11}$ are arbitrary locally analytic functions of $z$.

Case II. $a_3 \neq 0, a_4 = 0$: If $y_0j, j = 1, 2$, are roots of (2.4.b), and $(r_{j1}, r_{j2})$ are the Fuchs indices corresponding to $y_0j$, then let

$$r_{j1}r_{j2} = P(y_0j) = p_j, \quad j = 1, 2, \quad (2.37)$$
where
\[
P(y_{0j}) = 3(6 - 2c_1 - c_2) + 2(a_1 + a_2)y_{0j} - a_3y_{0j}^3, \quad j = 1, 2,
\]
and \( p_j \in \mathbb{Z} \setminus \{0\} \). In order to have a principal branch, at least one of the \( p_j \) should be a positive integer. Equation (2.4.b) gives
\[
a_3 = -\frac{6 - 2c_1 - c_2}{y_{01}y_{02}}, \quad 2a_1 + a_2 = a_3(y_{01} + y_{02}). \quad (2.39)
\]
Then (2.38) can be written as
\[
P(y_{01}) = (6 - 2c_1 - c_2) \left(1 - \frac{y_{01}}{y_{02}}\right),
\]
\[
P(y_{02}) = (6 - 2c_1 - c_2) \left(1 - \frac{y_{02}}{y_{01}}\right). \quad (2.40)
\]
If \( p_1p_2 \neq 0 \) and \( 6 - 2c_1 - c_2 \neq 0 \), then \( p_j \) satisfy the following hyperbolic type of Diophantine equation
\[
\frac{1}{p_1} + \frac{1}{p_2} = \frac{n^2}{6 - 2c_1 - c_2}. \quad (2.41)
\]
For each solution set \((p_1, p_2)\) of (2.41) one should find \((r_{ji}, r_{j2})\) such that \( r_{ji}, i = 1, 2 \) are distinct integers and \( r_{j1} + r_{j2} = p_j \). Then \( y_{0j} \) and \( a_i \) can be obtained from (2.39), (2.40) and \( r_{j1} + r_{j2} = a_1y_{0j} - c_1 + 7 \).

When \((c_1, c_2) = (3, -2 + 2/n^2)\), the Diophantine equation (2.41) takes the form
\[
\frac{1}{p_1} + \frac{1}{p_2} = \frac{n^2}{2(n^2 - 1)} \quad n \neq \pm 1. \quad (2.42)
\]
The general solution of (2.42) is
\[
p_1 = 2(n^2 - 1) + d_i, 
\]
\[
p_2 = \frac{2(n^2 - 1)^2}{n^2} \left[1 + \frac{2(n^2 - 1)}{d_i}\right], \quad n \neq 0, \quad (2.43)
\]
where \( \{d_i\} \) is the set of divisors of \( 4(n^2 - 1)^2 \neq 0 \). When \( n = \pm 3 \), (2.43) gives \((p_1, p_2) = (2, 16)\), which does not lead any Fuchs indices. \((p_1, p_2) = (1, -3), (2, 6), (3, 3), \) when \( n = \pm 2 \). We have distinct Fuchs indices for both branches only for \((p_1, p_2) = (2, 6), (3, 3)\). If \((p_1, p_2) = (2, 6)\), we have
\[
y_{01} = -\frac{1}{a_1} : \quad (r_{11}, r_{12}) = (1, 2), \quad \text{and} \quad y_{02} = \frac{3}{a_1} : \quad (r_{21}, r_{22}) = (1, 6). \quad (2.44.a, b, c)
\]
\[
y'' = 3\frac{y'y''}{y} - 3(\frac{y'}{y})^3 + a_3y^2y', \quad (2.45.a, b)
\]
\[
y'' = 3\frac{y'y''}{y} - 3(\frac{y'}{y})^3 + a_3y^2y', \quad (2.45.c)
\]
which is of Painlevé type [14].

After adding the non-dominant terms \( F_2 \) given in (2.2.b) to (2.45), the leading order \( \tilde{a} \) of \( u \) as \( z \to 0 \) is \( \tilde{a} = -1 \) and \( (\tilde{r}_{j1}, \tilde{r}_{j2}) = (1, 3) \), \( j = 1, 2 \), if \( A_{12} \neq 0 \), \( A_5 = A_6 = A_9 = A_{15} = 0 \) and \( A_{12}(z_0)u_0^2 = 3/2 \). Then, we have the equation
\[
y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + k_1y + k_2y^2y', \quad (2.47)
\]
where \( k_1, k_2 \) are constants. Integration of (2.47) once yields
\[
y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + k_1y + k_2y^2y', \quad (2.48)
\]
(2.48) is of Painlevé type [14].

When \((c_1, c_2) = (3 - 1/n, -2 + 1/n + 1/n^2)\), the general solution of the Diophantine equation (2.41) is
\[
\frac{p_1}{n^2} = \frac{2n^2 + n - 1 + d_i}{n^2}, 
\]
\[
\frac{p_2}{n^2} = \frac{2n^2 + n - 1}{n^2} \left[1 + \frac{2n^2 + n - 1}{d_i}\right], \quad n \neq 0, \quad (2.49)
\]
where \( \{d_i\} \) is the set of divisors of \( 2(n^2 + n - 1)^2 \neq 0 \). When \( n = 1 \), \((p_1, p_2) = (1, -2), (3, 6), (4, 4)\). Only the solutions \((3, 6)\) and \((4, 4)\) give distinct Fuchs indices.
for both branches. The Fuchs indices and the simplified equations for these cases are as follows:

For \( (p_1, p_2) = (3, 6), \)

\[
y_0 = -\frac{1}{a_1} : \quad (r_{11}, r_{12}) = (1, 3),
\]

\[
y_2 = \frac{2}{a_1} : \quad (r_{21}, r_{22}) = (1, 6), \quad (2.50.a,b,c)
\]

\[
y'' = 2y'y'' + a_1 [y''' - (y')^2 + a_1y^2y'].
\]

(2.50,c) does not pass the Painlevé test, since the compatibility condition at \( r_{12} = 3 \) is not satisfied identically.

For \( (p_1, p_2) = (4, 4), \)

\[
y_0 = \frac{2}{a_3}, y_2 = -y_0 : \quad (r_{j1}, r_{j2}) = (1, 4), \quad j = 1, 2,
\]

\[
y'' = 2y'y'' + a_3y^3y', \quad (2.51.a,b)
\]

(2.51.b) was also considered in [12]. Integrating (2.51.b) once yields,

\[
y'' = a_3y^3 + k_1y^2, \quad k_1 = \text{constant}. \quad (2.52)
\]

(2.52) is of Painlevé type [14].

If we add the non dominant terms, then the leading order of \( u \) as \( z \to z_0 \) is \( \alpha = -1 \) and \( (r_1, r_2) = (0, 3) \) when \( A_5 = 0 \). The canonical form of the equation is as follows:

\[
yy'' = 2y'y'' + 2y^3y' + k_1yy', \quad k_1 = \text{constant}. \quad (2.53)
\]

(2.53) was also given in [11]. Integration of (2.53) once gives

\[
y'' = 2y^3 + k_2y^2 - \frac{k_1}{2}, \quad k_2 = \text{constant}. \quad (2.54)
\]

(2.54) is solvable by means of the elliptic functions.

When \( n = \pm 2, \pm 3 \), the solutions of the Diophantine equation (2.49) do not give any Fuchs indices.

When \( (c_1, c_2) = (3 - 3/n, -2 + 3/n - 1/n^2) \), the general solution of the Diophantine equation (2.41) is

\[
p_1 = \frac{2n^2 + 3n + 1 + d_i}{n^2}, \quad p_2 = \frac{2n^2 + 3n + 1}{n^2} \left[ 1 + \frac{2n^2 + 3n + 1}{d_i} \right], \quad n \neq 0, \quad (2.55)
\]

where \( \{d_i\} \) is the set of divisors of \( (2n^2 + 3n + 1)^2 \neq 0 \). It should be noted that, \( c_1 = c_2 = 0 \) when \( n = 1 \). For \( n = 2 \) we have \( (p_1, p_2) = (3, -15) \), \( (4, 60) \), \( (5, 15) \), \( (6, 10) \), but only \( (p_1, p_2) = (3, -15) \) gives the distinct Fuchs indices for both branches.

The Fuchs indices and the simplified equation of this case are

\[
y_0 = -\frac{3}{2a_1} : \quad (r_{11}, r_{12}) = (1, 3),
\]

\[
y_2 = -\frac{15}{4a_1} : \quad (r_{21}, r_{22}) = (-5, 3), \quad (2.56)
\]

\[
y'' = 3y'y'' - \frac{3}{4} (y')^3 + a_1 [yy'' + (y')^2] - \frac{1}{3} a_1^2 y^2 y'.
\]

Without loss of generality, one can set \( a_1 = 3/2 \), then integrating (2.56) once yields

\[
y'' = \frac{3}{4} (y')^2 + \frac{3}{4} y^2 - \frac{1}{4} y^3 + k_1, \quad (2.57)
\]

\( k_1 = \text{constant}. \)

This case was also given in [12], and (2.57) is of Painlevé type [3, 14].

If one adds the non-dominant terms, then \( \alpha = -2 \) and \( (r_1, r_2) = (0, 1) \). The transformation (2.5), the compatibility conditions at \( (r_{11}, r_{12}) = (1, 3), \) \( r_{22} = 3 \) and the compatibility conditions at \( r_2 = 1 \) allow one to determine all the coefficients \( A_k \). Hence,

\[
y^2y'' = \frac{3}{2} y'y'y'' - \frac{3}{4} (y')^3 - \frac{3}{2} [y^3y'' + y^2(y')^2]
\]

\[
- \frac{3}{4} y^4y' + A_7 y^2y' + A_7'y^3, \quad (2.58)
\]

where \( A_7 \) is an arbitrary analytic function of \( z \). Integration of (2.58) once yields

\[
y'' = \frac{3}{4} (y')^2 - \frac{3}{2} yy'' - \frac{1}{4} y^3 + A_7 y + k_1, \quad (2.59)
\]

where \( k_1 \) is an integration constant. (2.59) possesses the Painlevé property [3, 14].

For \( n = -3, -2 \), \( (p_1, p_2) = (1, -10) \) and \( (p_1, p_2) = (1, 3) \), respectively. But for both cases there are double Fuchs index at \( \pm 1 \). For \( n = 3 \), the only solution of (2.55) is \( (p_1, p_2) = (4, 14) \). This solution gives the Fuchs indices \( (r_{11}, r_{12}) = (1, 4) \) for the first branch, but no Fuchs indices for the second branch.
When \((c_1, c_2) = (3 - 2/n, -2 + 2/n)\), the general solution of Diophantine equation (2.41) is
\[
p_1 = \frac{2(n + 1) + d_i}{n},
\]
\[
p_2 = \frac{2(n + 1)}{n} \left[ 1 + \frac{2(n + 1)}{d_i} \right], \quad n \neq 0,
\]
where \(\{d_i\}\) is the set of divisors of \(4(n + 1)^2 \neq 0\).
\((p_1, p_2) = (2, -2(n + 1))\) is a particular solution of the Diophantine equation which corresponds to \(d_i = 2\).
The Fuchs indices and the simplified equation corresponding to this case are as follows:
\[
y_{01} = \frac{n + 2}{n a_1}; \quad (r_{11}, r_{12}) = (1, 2),
\]
\[
y_{02} = -\frac{(n + 1)(n + 2)}{n a_1}; \quad (r_{21}, r_{22}) = (-1 + n, 2),
\]
\[
y''' = \left(3 - \frac{2}{n}\right) \frac{y''}{y} - \left(2 - \frac{2}{n}\right) \left(\frac{y'}{y}\right)^3 + \frac{2n}{(n + 2)^2} a_1 y' y', \quad n \neq 0, -1, -3
\]
Without loss of generality, we can set \(a_1 = 1 + 2/n\). If one lets \(y = u'/u\), and then \(u' = v''\), the last equation (2.61.c) yields
\[
v''' = v'''.
\]
Integrating (2.62) once gives a linear equation for \(v\).
Therefore (2.61) is of Painlevé type and was also considered in [12].

In particular, for \(n = -2\), (2.60) implies that \((p_1, p_2) = (2, 2)\). Then \(y_{0j}\), the Fuchs indices for both branches and the simplified equation are as follows [12]:
\[
y_{01} = \frac{1}{a_3},
\]
\[
y_{02} = -y_{01}; \quad (r_{j1}, r_{j2}) = (1, 2), \quad j = 1, 2,
\]
\[
y''' = 4 \frac{y'' y''}{y} - 3 \left(\frac{y'}{y}\right)^3 + a_3 y^3 y'.
\]
Integrating (2.63) once yields
\[
y' = \frac{(y''^2}{y} + a_3 y^3 + k_1 y^2, \quad k_1 = \text{constant}
\]
(2.64) is of Painlevé type [14].

After adding the non-dominant terms, one finds the following canonical form of the equations. If \(A_5 \neq 0\), \(A_6 = -3A_5\) and \(A_9 = A_{12} = A_{15} = 0\), then \(\alpha = -1\), \(A_5(\zeta_0)u_0 = -1\) and \((\hat{r}_{11}, \hat{r}_{12}) = (1, 3)\). The canonical form of the equation is
\[
y^2 y''' = 4yy'' - 3(y')^3 + y^4 y' + A_5 \left[ yy'' - 3(y')^2 + y^4 \right]
\]
\[
+ 3A_5yy' - A_5'^2 y^2,
\]
where \(A_5\) is a locally analytic arbitrary function of \(\zeta\). If \(A_{12} \neq 0\), and \(A_5 = A_6 = A_9 = A_{15} = 0\), then \(\alpha = -1\), \(A_{12}(\zeta_0)\zeta_0^2 = 3\) and \((\hat{r}_{11}, \hat{r}_{12}) = (2, 3), \ j = 1, 2\). The compatibility conditions at the Fuchs indices give
\[
A_{10}' = \frac{3}{2} A_{12} A_{10}' + \frac{1}{2} \left(\frac{A_{12}'}{A_{12}}\right)^2 A_{10} = 0,
\]
\[
A_{11} = -\frac{3}{4} A_{12} A_{10} A_{10}' + \frac{3}{8} \left(\frac{A_{12}' A_{10}}{A_{12} A_{10}}\right).
\]
\[
A_{12} A_{10}' = (A_{12}')^2, \quad A_{14} = -\frac{1}{3} A_{12}'
\]
\[
A_{13} = -A_{10}' + \frac{A_{12}'}{4A_{12}} A_{10}.
\]

Therefore, if \(A_{12} = k_1 = \text{constant} \neq 0\), then the canonical form of the equation is
\[
y^2 y''' = 4yy'' - 3(y')^3 + y^4 y' + (k_2 + k_3 \zeta)y' + k_1 y'
\]
\[
- \frac{3}{4} \frac{k_3}{k_1} (k_2 + k_3 \zeta)^3 - k_3 \zeta^2.
\]
where \(k_2\) and \(k_3\) are constants. If \(A_{12} = k_2 e^{k_i \zeta}, k_1 k_2 \neq 0\), then the canonical form of the equation is
\[
y^2 y''' = 4yy'' - 3(y')^3 + y^4 y' + \left( k_3 e^{k_i \zeta} + k_4 e^{k_i \zeta/2} \right) y y'
\]
\[
+ k_2 e^{k_i \zeta} y' - \frac{3}{4} k_1 k_3 \left( k_3 e^{k_i \zeta} + k_4 e^{k_i \zeta/2} \right) y^3
\]
\[
- \frac{1}{4} k_1 \left( 3k_3 e^{k_i \zeta} + k_4 e^{k_i \zeta/2} \right) y^2 - \frac{1}{3} k_1 k_2 e^{k_i \zeta} y.
\]
(r_{21}, r_{22}) = (-3, 2). The canonical form of the equation in this case is
\[ 2y''' = 4yy'' - 3(y')^3 + y^4y' + A5[y'y'' - 2(y')^2] \quad (2.69) \]
\[
+ \frac{3}{2} A_5 y' + A_{11} y^3 - \frac{1}{4} A_{12} y^2 - \frac{1}{2} A_{13} y + \frac{1}{4} A_{14} y, 
\]
where \(A_5, A_{11}\) are arbitrary locally analytic functions of \(z\). Similarly, for \(n = 1\) one can obtain the following canonical form of the equations, such that the corresponding simplified equation is not contained in (2.61c):
\[
yy''' = y'y'' + 4y^3y' + k_1 y^2 y' \\
\quad - \left(\frac{k_1 k_2}{6} - k_3\right)y' + k_2 y^2 + \frac{k_1 k_2}{6}, \quad (2.70) \\
yy''' = y'y'' + 4y^3y' + \frac{1}{z} \left[ y'y'' - 2(y')^2 \right] + \frac{k_1}{z} y^2 y' - \frac{2k_1}{z^2} y^3 \\
\quad - \left(\frac{k_2}{2z^2} - \frac{k_1}{z} \right) y^2 + \left(\frac{k_1}{10z^3} + \frac{k_1 k_2}{6z} + k_3\right)y' \\
\quad + \left(\frac{4k_1}{z^2} - \frac{k_1 k_2}{27z^2} + \frac{k_1 k_2}{3z^2} + k_3\right)y, \quad (2.71) \\
yy''' = y'y'' + 4y^3y' - \frac{1}{z} \left[ 3y'y'' - 2(y')^2 - k_1 z^2 y^3 \right] \\
\quad - 4y^4 - \frac{8}{3} k_1 z^3 y' + \left(\frac{k_2}{2z} + \frac{k_2}{z}\right)^2 y, \quad (2.72) \\
yy''' = \left(\frac{k_1}{144} z^3 + \frac{k_1 k_2}{12 z} - \frac{k_1}{z}\right)y' + \left(\frac{k_1}{36 z^2} + \frac{k_1 k_2}{6 z} + k_3\right), 
\]
when \((A_1, A_2) = (0, 0), (A_1, A_2) = (1/z, 0)\) and \((A_1, A_2) = (A_5 - A_{12}^2)/A_2, 1/z\), respectively, where \(k_i\) are constants. Integration of (2.70) and (2.71) yields
\[
y'' = 2y^3 + k_1 y^2 + (k_2 z + k_4) y + \frac{k_1 k_2}{6} z + k_3, \quad (2.73) \\
v'' = 2v^3 + (k_4 z - k_2) v - (k_3 + \frac{k_1 k_2}{6}), \quad (2.74) 
\]
respectively, where \(k_4\) is an integration constant and \(v = y + (k_1/6z)\) in (2.74).

**Case III.** \(a_j \neq 0\) : In this case there are three branches corresponding to roots \(y_{0j}, \ j = 1, 2, 3,\) of (2.4.b). Equation (2.4.b) implies that
\[
\prod_{j=1}^{3} y_{0j} = -\frac{6 - 2c_1 - c_2}{a_4}, \quad (2.75) \\
\]
\[
\prod_{i \neq j} y_{0i} y_{0j} = \frac{1}{a_4} (2a_1 + a_2), \quad \sum_{j=1}^{3} y_{0j} = \frac{a_3}{a_4}. \quad (2.75) 
\]
Let
\[
P(y_{0j}) = 3(6 - 2c_1 - c_2) + 2(2a_1 + a_2) y_{0j} \\
- a_3 y_{0j}^2, \quad j = 1, 2, 3. \quad (2.76)
\]
If the Fuchs indices (except \(r_{j0} = -1\)) are \(r_{ji}, i = 1, 2,\) corresponding to \(y_{0j}\), then (2.4.a) implies that
\[
\prod_{i=1}^{2} r_{ji} = P(y_{0j}) = p_j. \quad (2.77)
\]
In order to have a principal branch, \(p_j\) should be integers such that at least one of them is positive. Equations (2.75) and (2.76) give
\[
p_j = (6 - 2c_1 - c_2) \prod_{l=1, l \neq j}^{3} \left(1 - \frac{y_{0l}}{y_{0j}}\right), \quad j = 1, 2, 3, \quad (2.78)
\]
and hence \(p_j\) satisfies the Diophantine equation
\[
\prod_{j=1}^{3} \frac{1}{p_j} = \frac{1}{6 - 2c_1 - c_2}. \quad (2.79)
\]
If \(\prod_{j=1}^{3} p_j \neq 0\) and \(6 - 2c_1 - c_2 \neq 0:\) from (2.78) one has the following system for \(y_{0j}^2:\)
\[
p_1 (y_{02} - y_{03}) = \mu y_{01}, \\
p_2 (y_{03} - y_{01}) = \mu y_{02}, \\
p_3 (y_{01} - y_{02}) = \mu y_{03},
\]
where
\[
\mu = \frac{6 - 2c_1 - c_2}{y_{01} y_{02} y_{03}}, \quad (y_{01} - y_{02})(y_{02} - y_{03})(y_{01} - y_{03}). \quad (2.81)
\]
On the other hand, (2.78) gives that
\[ \prod_{j=1}^{3} p_j = -(6-2c_1-c_2)^2. \]
(2.82)

Then, if \( a_1 \neq 0 \) (note that \( r_{j1}+r_{j2} = a_1y_{0j} - c_1 + 7 \)
then \((6-2c_1-c_2)^2 \mu > 0 \) and a real number. Therefore, \( \prod_{j=1}^{3} p_j < 0 \). That is, if \( p_1 > 0 \), then either \( p_2 \) or \( p_3 \) is a negative integer. So one should consider cases \( a_1 = 0 \) and \( a_1 \neq 0 \) separately.

**III.A. \( a_1 = 0 \):** From (2.4.a), one has
\[ r_{j1} + r_{j2} = 7 - c_1. \]
(2.83)

Thus \( c_1 \) is an integer. Since
\[(r_{j1} - r_{j2})^2 = (r_{j1} + r_{j2})^2 - 4r_{j1}r_{j2}, \]
(2.84)

\((7-c_1)^2 - 4p_j \) is a perfect square. Then for each five cases one can determine \( p_j \). By using the system (2.80) and (2.75), one obtains \( y_{0j} \) and \( a_{mj} \), \( m = 2, 3, 4 \).

When \((c_1, c_2) = (3, -2 + \frac{2}{n}) \), since \( c_1 = 3 \), (2.84) and (2.83) give
\[(r_{j1} + r_{j2})^2 = 16 - 4p_j, \quad j = 1, 2, 3. \]
(2.85)

So \( 16 - 4p_2 \) must be a perfect square. If we let \( p_1, p_2 > 0 \), then (2.85) implies that \( p_2 = 3 \). The Diophantine equation (2.79) implies that \( p_2 \) is an integer when \( n = \pm 1 \). But \( 6 - 2c_1 - c_2 = 0 \) when \( n = \mp 1 \).

When \((c_1, c_2) = (3 - \frac{1}{n}, -2 + \frac{1}{n} + \frac{2}{n}) \), \( c_1 \) is an integer and \( 6 - 2c_1 - c_2 \neq 0 \) only if \( n = 1 \). The Fuchs indices and the simplified equation for this case are [12]
\[ y_{0j} = -\frac{2}{a_4}, \quad (r_{j1}, r_{j2}) = (2, 3), \quad j = 1, 2, 3, \]
\[ y'' = 2\frac{y'y''}{y} + a_4y^4. \]
(2.86.a,b)

If we add the non-dominant terms to (2.86), then \( a_{1} = -1, a_{0} = \) arbitrary \( \neq 0 \) and the Fuchs indices are \((\hat{r}_1, \hat{r}_2) = (0, 3) \). The transformation (2.5), the compatibility conditions at \((r_{j1}, r_{j2}) \), \( j = 1, 2, 3 \), and at \((\hat{r}_1, \hat{r}_2) \) imply that \( A_k = 0, k = 1, \ldots, 11 \). So the canonical form of the equation is the simplified equation (2.86.b).

When \((c_1, c_2) = (3 - \frac{3}{n}, -2 + \frac{2}{n} - \frac{2}{n^2}) \), \( c_1 \in Z \) implies that \( n = \pm 1, \pm 3 \). But only for \( n = -3, 6 - 2c_1 - c_2 \neq 0, c_1^2 + c_2^2 \neq 0 \) and the Fuchs indices are distinct for all three branches. The indices and the simplified equation for this case are
\[ y_{01} = -\frac{1}{3a_2}, \quad (r_{11}, r_{12}) = (1, 2), \]
\[ y'' = \frac{2}{3a_2} : (r_{21}, r_{22}) = (1, 2), \]
\[ y_{03} = \frac{5}{3a_2} : (r_{31}, r_{32}) = (-2, 3, 5, 2.87.a-d) \]
\[ y'' = \frac{4}{9} \frac{y'y''}{y} - \frac{28}{9} \frac{(y'y')^3}{y^3} + a_2 \left[ (y'y')^2 + 6a_2y^2y'y'' + 3a_2^2y''^2 \right]. \]
(2.87.d) does not pass the Painlevé test, since the compatibility conditions are not satisfied identically.

When \((c_1, c_2) = (3 - \frac{2}{n}, -2 + \frac{2}{n}) \), \( c_1 \in Z \) implies that \( n = \pm 1, \pm 2 \). For these values of \( n \), there are no distinct Fuchs indices for all branches.

When \((c_1, c_2) = (3, -2) \), the solutions of the Diophantine equation (2.79) do not give distinct Fuchs indices.

**III.B. \( a_1 \neq 0 \):** Once the solution set \( p_j = r_{j1}r_{j2}, \quad j = 1, 2, 3, \) of (2.79) is known, \( y_{0j} \) and \( a_{i} \), \( i = 1, 2, 3, 4 \), can be determined from the equations (2.80), (2.75) and
\[ r_{j1} + r_{j2} = a_1y_{0j} + 7 - c_1, \quad j = 1, 2, 3. \]
(2.88)

When \((c_1, c_2) = (3, -2 + \frac{2}{n}) \), \( (p_1, p_2, p_3) = (2, 4(n-1), -4(n+1)) \) is a particular solution of the Diophantine equation (2.79), and \( \mu = \pm 4n \). The Fuchs indices are distinct only for \( \mu = -4n \). The indices and the simplified equation for this case are
\[ y_{01} = -\frac{1}{a_1} : (r_{11}, r_{12}) = (1, 2), \]
\[ y_{02} = \frac{n-1}{a_1} : (r_{21}, r_{22}) = (4, n-1), \]
\[ y_{03} = \frac{n+1}{a_1} : (r_{31}, r_{32}) = (4, -(n+1)), \]
\[ y'' = \frac{3}{n} \frac{y'y''}{y} - \frac{2}{n^2} \frac{(y'y')^3}{y^3} \]
\[ + a_1 \left[ (yy'y''y''')^2 + \frac{6}{n^2}a_1y'y''y'' + \frac{2}{n^2}a_1y''^2 \right], \]
\[ n \neq 0, \pm 1, \pm 5. \]
(2.89.a-d)
\[(2.89.d) \text{ was also considered in } [12]. \text{ If one lets } y = u'/u \text{ and } u'' = v'' \text{ then } (2.89.d) \text{ yields} \]
\[ vv'' = 3y'v''. \]
(2.90)

Integrating (2.90) once gives \( v'' = k_1v^3 \), \( k_1 \) constant.

If \( k_1 = 0 \), then \( v = k_2z + k_3, \quad k_1 \) constant, \( i = 2, 3 \). If
$k_1 \neq 0$, then $v = \sum_{i=0}^{\infty} v_i (z - z_0)^{i+1}$, where $z_0$ is arbitrary. Since $u' = v^2$, in order to $u$, and consequently $y$, being single valued, it is necessary and sufficient that $u'$ does not contain the term $(z - z_0)^{-1}$. That is $n \neq 0, \pm(1+4m)$, where $m \in \mathbb{Z}$.

Particularly for $n = 2$, after adding the non-dominant terms to (2.89), $\alpha = -1$ is the possible leading order of $u = 1/y$ as $z \to z_0$ if $A_{12} \neq 0$, $A_5 = A_6 = A_9 = A_{15} = 0$ and $A_{12}(z_0)u_0^2 = 3/2$. The Fuchs indices are $(\tau_{j1}, \tau_{j2}) = (1, 3)$, $j = 1, 2$ and the canonical form of the equation is

\[
y'^2 = 3y' + \left(\frac{3}{2}y''^2 - \frac{3}{2}y'^2 + \frac{3}{2}y^2y'\right) + \frac{1}{y^6} + \left(\frac{k_1}{3}z^2 + k_2z + k_3\right) \left(y^3y' + y^4\right) - \left(\frac{2k_1}{3}z + k_2\right)y^3 + k_1y'^2 + \frac{k_1}{3}y',
\]

(2.91)

where $k_i$ = constant, $i = 1, 2, 3$.

When $(c_1, c_2) = (3 - \frac{1}{2}, -2 + \frac{1}{2} + \frac{1}{y})$, $(p_1, p_2, p_3) = (2, 6(2n - 1), -3(n + 1))$ is a particular solution of (2.79). Then the system (2.80) has non-trivial solution if $\mu = \pm 6n$. For both values of $\mu$, we have the following simplified equations:

\[
y_{01} = -\frac{n + 1}{na_1} : (r_{11}, r_{12}) = (1, 2),
\]

\[
y_{02} = -\frac{(n + 1)^2}{na_1} : (r_{21}, r_{22}) = (3, -(n + 1)),
\]

\[
y_{03} = \frac{(n + 1)(2n - 1)}{na_1} : (r_{31}, r_{32}) = (6, 2n - 1)
\]

\[
y''' = \left(3 - \frac{1}{n}\right)\frac{y'''}{y} - \left(2 - \frac{1}{n} - \frac{1}{n^2}\right)\left(\frac{y'^3}{y^2}\right) + \frac{a_1}{n+1} \left[y'' + \frac{3}{n(n+1)}(y')^2 + \frac{3-n}{(n+1)^2}a_1y^2y'\right] - \frac{n}{(n+1)^2}a_1^2y^4, n \neq 0, -1, -4.
\]

(2.92) was also considered in [12]. Substitution of $y = u'/u$ in (2.92) and then letting $u' = v^2$ give the following equation for $v$

\[
v'' = 2v'v''
\]

(2.93)

Integration of (2.93) once gives $v'' = kv_1^2$, $k_1$ = constant. If $k_1 = 0$ then $v = k_2z + k_3$, $k_i$ = constants $i = 2, 3$. If $k_1 \neq 0$, then $v = \sum_{i=0}^{\infty} v_i (z - z_0)^{i+2}$, $z_0$ = arbitrary. Therefore, if $n \neq -3m - 1$, $m = 0, 1, 2, \ldots$, $u$ and consequently $y$ is a single valued function of $z$.

In particular for $n = 1$, $(p_{11}, p_{22}, p_{33}) = (3, -30), (2, N, -N), N \in \mathbb{Z}$, exist solutions of (2.79). For $(p_{11}, p_{22}, p_{33}) = (3, -30)$, the system (2.80) has a non-trivial solution if $\mu = \pm 15$. Only the $\mu = -15$ case gives distinct Fuchs indices for all branches. The simplified equations for this case are as follows:

\[
y_{01} = -\frac{1}{a_1} : (r_{11}, r_{12}) = (1, 3),
\]

\[
y_{02} = \frac{1}{a_1} : (r_{21}, r_{22}) = (1, 5),
\]

\[
y_{03} = -\frac{4}{a_1} : (r_{31}, r_{32}) = (-5, 6), (2.94.a-d)
\]

\[
y''' = 2y'y'' - a_1 \left[3y'' - \frac{3}{2}(y')^2 + 2ay^3y' - \frac{1}{2}a_1^2y^4\right].
\]

Letting $a_1 = -1$, integrating (2.94.d) once yields

\[
y''' = \frac{3}{2}(y')^2 - \frac{1}{2}y^3 + k_1, k_1 = constant,
\]

(2.95)

which is solvable by means of elliptic functions [14]. After adding the non-dominant terms $\alpha = -1$ if $A_5 = 0$, the indices are $(\tau_1, \tau_2) = (0, 3)$. Then the canonical form of the equation is

\[
y''' = y'y'' - \frac{3}{2}(y')^2 + \frac{3}{2}y'^2 + 2y^3y' + \frac{1}{2}y^5 + k_1y,
\]

(2.96)

$k_1$ = constant.

For $(p_{11}, p_{22}, p_{33}) = (2, N, -N)$, (2.80) implies that $\mu = \pm N$. For $\mu = N$, $y_{01} = 0$, and $\mu = -N$ we have the following equation:

\[
y''' = 2y'y'' + a_1 \left[4y'^2 + \frac{N^2 + 12}{4 - N^2}(y')^2 - \frac{16}{4 - N^2}a_1y^2y'\right] + \frac{4}{4 - N^2a_1^2y^4}, N \neq \pm 2,
\]

(2.97)

with $a_1y_{01} = -2$, $a_1y_{02} = (N - 2)/2$, $a_1y_{03} = -(N + 2)/2$, and $(\tau_{11}, \tau_{12}) = (1, 2)$. The Fuchs indices for the second and the third branches satisfy the following equations, respectively,

\[
r_{21}' = \frac{N + 8}{2}r_{21} + N = 0,
\]

\[
r_{31}' + \frac{8 - N}{2}r_{31} - N = 0
\]

(2.98)
The compatibility condition at \( r_{12} = 2 \) is not satisfied identically unless \( N = 6 \). Then from (2.98), the indices are \((r_{21}, r_{22}) = (1, 6)\) and \((r_{31}, r_{32}) = (-2, 3)\). The corresponding simplified equation is (2.92) for \( n = 1 \). If one adds the non-dominant terms then, \((\hat{r}_1, \hat{r}_2) = (0, 3)\). The transformation (2.5), the compatibility conditions at \((r_{11}, r_{12}) = (1, 2)\), \((r_{21}, r_{22}) = (1, 6)\), \(r_{22} = 3\) of \( y \) and the compatibility conditions at the Fuchs indices of \( u \) imply that all the coefficients \( A_k \) are zero except \( A_{10} = k_1 = \text{constant} \). So, the canonical form of the equation is

\[
y^{m'} = 2y'y'' - 2y^2y''' + 3y(y')^2 + 2y^3y' + y^5 + k_1y, \quad (2.100.a-d)
\]

When \((c_1, c_2) = (3 - 3/n, -2 + 3/n - 1/n^2)\), \((p_1, p_2, p_3) = (2, 2(2n + 1), -(n + 1))\) is a particular solution of (2.79). For these values of \( p_j \), the system (2.80) has a nontrivial solution if \( \mu = \pm 2n \). Only for \( \mu = 2n \) there are distinct indices for all three branches. The indices and the corresponding simplified equation are

\[
y_0 = \frac{n + 3}{na_1} : (r_{11}, r_{12}) = (1, 2), \quad (2.100.a-d)
\]

\[
y_0 = \frac{(n + 3)(2n + 1)}{na_1} : (r_{21}, r_{22}) = (-2 + n, -2), \quad \n_0 = \frac{(n + 3)(n + 1)}{na_1} : (r_{31}, r_{32}) = (-n + 1, 1),
\]

\[
y'' = 3(1 - \frac{1}{n}) \frac{y'y''}{y} - \left(2 - \frac{3}{n} + \frac{1}{n^2}\right) \frac{(y')^3}{y^2}
+ a_1 \left[ny'' + \frac{3}{n + 3} (y')^2 = \frac{3(n + 1)}{(n + 3)^2} a_1 y^2y' + \frac{n}{(n + 3)^2} a_1^2 y^4 \right], \quad n \neq -1, -2, -3.
\]

(2.100.d) was also considered in [12]. Substituting \( y = u'/u \) in (2.100) and letting \( u' = v'' \) gives

\[
v'' = \frac{3}{2} y'' y' - \frac{3}{4} (y')^3 - \frac{5}{2} y^3 y'' - \frac{3}{4} y^2 (y')' - \frac{9}{4} y^4
\]

\[
- \frac{1}{4} y^6 + A_7 y^2 y' + y^4 + A_{11} y^3, \quad (2.102)
\]

where \( A_7, A_{11} \) are arbitrary locally analytic functions of \( z \).

When \((c_1, c_2) = (3 - 2/n, -2 + 2/n)\), and \( n = 1 \) the solutions of the Diophantine equation (2.79) are \((p_1, p_2, p_3) = (3, 24, -8)\), \((5, 16, -80)\), \((5, 19, -380)\), \((6, 10, -60)\), \((7, 8, -56)\), \((4, -N, N)\), \(N \in \mathbb{Z}_+\). Only for \((3, 24, -8)\) and \((4, -N, N)\) there are distinct Fuchs indices for all branches. The indices and the simplified equations for these cases are as follows: For \((p_1, p_2, p_3) = (3, 24, -8)\):

\[
y_{10} = -\frac{2}{a_1} : (r_{11}, r_{12}) = (1, 3),
\]

\[
y_{02} = \frac{4}{a_1} : (r_{21}, r_{22}) = (4, 6),
\]

\[
y_{03} = -\frac{4}{a_1} : (r_{31}, r_{32}) = (-2, 4), \quad (2.103.a-d)
\]

\[
y'' = \frac{y'y''}{y} + a_1 \left( y'' + \frac{1}{4} a_1 y^2 y' - \frac{1}{8} a_1^2 y^4 \right).
\]

This case was considered in [12]. After adding the non-dominant terms, if \( A_5 = 0 \), then \( \hat{p} = -1 \) and the Fuchs indices are \((\hat{r}_1, \hat{r}_2) = (0, 2)\). The transformation (2.5) and the compatibility conditions at \((r_{11}, r_{12}) = (1, 3)\), \((r_{21}, r_{22}) = (4, 6)\) and at \( r_2 = 2 \) imply that \( A_m = 0, m = 1, 2, \ldots, 6 \) and

\[
A_7 = A_8 = A_9 = k_1, \quad A_0 = k_1, \quad A_{10} = -A_9^2,
\]

where \( k_1 \) is a constant. It should be noted that the equation for \( A_7 \) is the autonomous part of the second member of the first Painlevé hierarchy [6, 8]. From (2.104) we have the following two cases: If \( k_1 = 0 \) and \( A_7 = -12/z^2 \) then

\[
y'' = y'y'' - 2y^2y'' + y^3y' + y^5 - \frac{12}{z^2} y^3 + \frac{24}{z^2} y' + \frac{72}{z^4} y, \quad (2.105)
\]
If \( A_2 = k_2 z + k_3, k_1 = \text{constant}, i = 2, 3 \), then the canonical form of the equation is
\[
y'''' = y'''' + 2y'' + y' + y^5 + (k_2 z + k_3)y^3 + k_2(y' + y^3).
\]
(2.106)

For \((p_1, p_2, p_3) = (4, -N, N)\): \( p_1 = 4 \) implies that \((r_{11}, r_{12}) = (1, 4)\) and hence \(a_1 y_{01} = 1\). By using the system (2.80), one finds \( y_{01} \) and \( y_{03} \) in terms of \( a_1 \) and \( N \). So, the Fuchs indices \( r_{22} \) and \( r_{30}, i = 1, 2 \), satisfy the equations
\[
\begin{align*}
r_{22}^2 - \frac{44 + N}{8} r_{22} + N &= 0, \\
r_{30}^2 - \frac{44 - N}{8} r_{30} - N &= 0,
\end{align*}
\]
(2.107)
respectively, and the simplified equation is
\[
y'''' = \frac{y''''}{y} + a_1 y'' + 2N^2 - 144 \frac{a_1(y')^2}{16 - N^2} + a_2 y' + a_3 y^3, \\
+ \frac{512}{16 - N^2} a_2 y' + \frac{256}{16 - N^2} a_3 y^3, N \neq \pm 4.
\]
(2.108)
The compatibility condition at \( r_{12} = 4 \) is not satisfied identically unless \( N = 12 \). Then, (2.107) gives that \((r_{21}, r_{23}) = (3, 4)\) and \((r_{31}, r_{32}) = (-2, 6)\), respectively. Thus we have the simplified equation [12]
\[
y'''' = \frac{y''''}{y} + a_1 y'' + 4a_1 y' + 2a_3 y^3.
\]
(2.109)
For this case, the canonical form of the equation is \( \alpha = -1 \), \((\tilde{r}_1, \tilde{r}_2) = (0, 2)\) and
\[
y'''' = y'''' - y^2 y'' + 4y^3 y' + 2y^5 + (2k_1 z + k_2) y^3 + k_1 (y' + y^3),
\]
(2.110)
where \( k_1, k_2 \) are constants.

When \((c_1, c_2) = (3, -2)\), the solutions of the Diophantine equation (2.79) do not lead to any distinct Fuchs indices.

3. Leading Order \( \alpha = -2 \)

\( \alpha = -2 \) is also a possible leading order of the equation (1.4). By adding the term \( y y' \), the following simplified equation with the leading order \( \alpha = -2 \), is obtained
\[
y'''' = c_1 y'''' + c_2 (y')^3 y'' + a y y',
\]
(3.1)
where \( a \) is constant and \( c_1, c_2 \) are given by (1.12), (1.15) and (1.16).

Substituting \( y = y_0 (z - z_0)^{-2} + \beta (z - z_0)^{-4} \) into (3.1) gives the following equations for the Fuchs indices \( r \) and \( y_{00} \), respectively:
\[
Q(r) = (r + 1)[r^2 + 2(c_1 - 5)r + 24 - 12c_1 - 8c_2] = 0,
\]
\[
a y_{00} = 12 - 6c_1 - 4c_2.
\]
(3.2.a
(3.2.b)
implies that there is only one branch. In order to have a principal branch, the indices \( r_1 \) and \( r_2 \) (except \( r_0 = -1 \)) should be distinct positive integers. Then (3.2.a) implies that \( 2c_1 \) and \( 4(3c_1 + 2c_2) \) should be integers.

To find the canonical forms of the equations, one should consider the following equations for \( c_2 = 0 \) and \( c_2 = 0 \), respectively:
\[
y'''' = c_1 y'''' + ay'' + A_{12} y'' + A_{2}(y')^2 + A_{3} y^3 + A_{4} y y'
\]
\[
+ A_{5} y'' + A_{6} y^2 + A_{7} y + A_{8},
\]
(3.3)
y'''' = c_1 y'''' + c_2 (y')^3 + ay'' + A_{12} y'' + A_{2}(y')^2
\]
\[
+ A_{3} y^3 + A_{4} y^2 y' + A_{5} y y' + A_6 (y')^2 + A_7 y^3
\]
\[
+ A_{8} y y' + A_{9} y y' + A_{10} y^2 + A_{11} y',
\]
(3.4)
where the \( A_k \) are locally analytic functions of \( z \). The coefficients \( A_k \) can be found by using the procedure described in the previous section.

When \((c_1, c_2) = (3, -2 + \frac{1}{n^2})\), the Fuchs indices satisfy \( r_1 + r_2 = 4 \) and \( r_1 r_2 = 4[1 - (4/n^2)] \). Hence, \( n = \pm 1, \pm 2, \pm 4 \), but \( n = \pm 1 \) does not lead a principal branch. Therefore we have the following cases: For \( n = \pm 2 \), the Fuchs indices, simplified equation and the canonical form of the equation are
\[
y_{00} = \text{arbitrary} : (r_1, r_2) = (0, 4),
\]
(3.5)
y'''' = \( 3 y'''' - \frac{3 (y')^3}{y^2} \),
\[
y'''' = \frac{3 y''''}{y} - \frac{3 (y')^3}{y^2} + \frac{1}{2} (k_1 z + k_3) y' + k_2,
\]
(3.6)
respectively, where \( k_i, i = 1, 2, 3 \) are constants.

For \( n = \pm 4 \):
\[
A y_{00} = \frac{3}{2} : (r_1, r_2) = (1, 3),
\]
y'''' = \( \frac{3 y''''}{y} - \frac{15 (y')^3}{8 y^2} + a y y' \),
(3.7.a,b)
Integration of (3.7.b) once yields
\[ v'' = \frac{1}{2} \left( \frac{y'}{v} \right)^2 + av^3 + k_1 v, \quad k_1 = \text{constant}, \quad (3.8) \]
where \( v^2 = y \). If we let \( a = 3/2 \), then (3.8) is of Painlevé type [14]. For this case, the canonical form of the equation is
\[ y''' = \frac{3}{2} \frac{\gamma''}{y} - \frac{15}{8} \left( \frac{y'}{y} \right)^3 + \frac{3}{2} yy' + k_1 y', \quad k_1 = \text{constant}. \quad (3.9) \]
Integration of (3.9) yields
\[ v'' = \frac{1}{2} \left( \frac{y'}{v} \right)^2 + 3v^2 - \frac{2k_1}{3} v + k_2 v^2, \quad (3.10) \]
where \( v^2 = y \) and \( k_2 \) is an integration constant. (3.10) is solvable by means of the elliptic functions [14].

When \((c_1, c_2) = (3 - 1/n, -2 + 1/n + 1/n^2), c_21 \) is an integer if \( n = 1, \pm 2 \). \( n = -1 \) does not lead a principal branch. So, when \( n = -2 \), we have the following simplified equation
\[ y_0 = \text{arbitrary} : (r_1, r_2) = (0, 3), \quad y''' = \frac{7}{2} \frac{\gamma''}{y} - \frac{9}{4} \left( \frac{y'}{y} \right)^3, \quad (3.11) \]
and the canonical form of the equation
\[ y''' = \frac{7}{2} \frac{\gamma''y''}{y} - \frac{9}{4} \left( \frac{y'}{y} \right)^3 + k_1 \frac{y'}{y}, \quad (3.12) \]
where \( k_1 \) is a constant. (3.12) yields
\[ u'' = \frac{5}{4} \frac{(u')^2}{u} + \frac{k_1}{2} u^2 + k_2, \quad k_2 = \text{constant} \quad (3.13) \]
after letting \( y = 1/u \) and integrating once. (3.13) is of Painlevé type [14]. When \( n = 1 \), we have
\[ y_0 = \text{arbitrary} : (r_1, r_2) = (0, 6), \quad y''' = 2 \frac{\gamma''y''}{y}, \quad (3.14) \]
The canonical form of the equations are as follows:
\[ y''' = 2 \frac{\gamma''y''}{y} + k_1, \quad k_1 = \text{constant} \quad (3.15) \]
\[ y''' = 2 \frac{\gamma''y''}{y} + (k_2 - 2k_1 c) \frac{y'}{y} + k_1, \quad k_1, k_2 = \text{constant} \quad (3.16) \]
For \( n = 2 \), the simplified equation is
\[ y_0 = \frac{2}{a} : (r_1, r_2) = (1, 4), \quad y''' = \frac{5}{2} \frac{\gamma''}{y} - \frac{5}{4} \left( \frac{y'}{y} \right)^3 + ay', \quad (3.17.a.b) \]
Integration of (3.17.b) once yields
\[ y''' = \frac{5}{4} \frac{(y')^2}{y} + \frac{a}{2} v^2 + k_1, \quad k_1 = \text{constant}. \quad (3.18) \]
(3.18) is solvable by means of elliptic functions. The canonical form of the equation is
\[ y''' = \frac{5}{2} \frac{\gamma''y''}{y} - \frac{5}{4} \left( \frac{y'}{y} \right)^3 + 2y'y' + k_1 y', \quad (3.19) \]
\[ k_1 = \text{constant}. \]
Integration of (3.19) yields
\[ v'' = \frac{3}{2} \left( \frac{y'}{v} \right)^2 + \frac{1}{2} \frac{\gamma''}{y} - \frac{k_1}{2} v + k_2 \frac{1}{2} v, \quad (3.20) \]
where \( v^2 = y \) and \( k_2 \) is an integration constant. (3.20) is solvable by means of elliptic functions.

When \((c_1, c_2) = (3 - \frac{2}{n}, -2 + 3/n - 1/n^2), c_21 \) is integer implies that \( n = \pm 1, \pm 2, \pm 3, \pm 6 \). If \( n = 1, -1 \) and \( n = \pm 3, \pm 6 \) then \( c_1 = c_2 = 0 \), there is no principal branch and there are no Fuchs indices respectively. Therefore, we have the following cases: For \( n = -2 \) the simplified equation is
\[ y_0 = \text{arbitrary} : (r_1, r_2) = (0, 1), \quad y''' = \frac{9}{2} \frac{\gamma''y''}{y} - \frac{15}{4} \left( \frac{y'}{y} \right)^3, \quad (3.21) \]
and the canonical form of the equation is
\[ y''' = \frac{9}{2} \frac{\gamma''y''}{y} - \frac{15}{4} \left( \frac{y'}{y} \right)^3 + k_1 y' + k_2 y', \quad (3.22) \]
where \( k_i, i = 1, 2 \) are constants. If we let \( y = 1/u \) in (3.22) and integrate once, then we have
\[ u'' = \frac{3}{4} \frac{(u')^2}{u} + \frac{k_2}{2} u^2 + k_1 u + k_3, \quad (3.23) \]
where \( k_3 \) is an integration constant and (3.23) is of Painlevé type [14]. For \( n = 2 \), the simplified equation is
\[ y_0 = \frac{6}{a} : (r_1, r_2) = (3, 4), \quad y''' = \frac{3}{2} \frac{\gamma''y''}{y} - \frac{3}{4} \left( \frac{y'}{y} \right)^3 + ay', \quad (3.24.a.b) \]
Letting \( a = 6 \) and integrating (3.24.b) once yields
\[
y'' = \frac{3}{4} \left( y' \right)^{2} y + 3y^{2} + k_{1}, \quad k_{1} = \text{constant.} \tag{3.25}
\]
(3.25) is of Painlevé type [3, 14]. The canonical form is
\[
y'' = \frac{3}{2} \frac{y'y''}{y} - \frac{3}{4} \left( \frac{y'}{y} \right)^{3} + 6yy',
\]
(3.26)
+ \( (k_{1}z + k_{2})y' + 2k_{1}y, \)

where \( k_{i} \) is constant, \( i = 1, 2. \)

When \((c_{1}, c_{2}) = (3 - 2/n, -2 + 2/n), 2c_{1} \) is integer if \( n = \pm 1, \pm 2, \pm 4. \) When \( n = -1, \) there is no principal branch. So, we have the following three cases:
For \( n = -4, \)
\[
y_{0} = \frac{1}{a} \cdot \, \, (r_{1}, r_{2}) = (1, 2), \tag{3.27.a,b}
y'' = \frac{7}{2} \frac{y'y''}{y} - \frac{5}{2} \left( \frac{y'}{y} \right)^{3} + ay',
\]
Setting \( a = 1 \) in (3.27.b) and integrating once yields
\[
y'' = \left( \frac{y'}{y} \right)^{2} + v^{3} + k_{1}v^{2}, \quad k_{1} = \text{constant}, \tag{3.28}
\]
where \( v^{2} = y, \) (3.28) is of Painlevé type [14]. The canonical form is
\[
y'' = \frac{7}{2} \frac{y'y''}{y} - \frac{5}{2} \left( \frac{y'}{y} \right)^{3} + \frac{y y'}{y} + k_{1} \frac{y'}{y}. \tag{3.29}
\]
Integration of (3.29) once yields
\[
y'' = \left( \frac{y'}{y} \right)^{2} + v^{2} - \frac{1}{3} \frac{k_{1}}{v} + \frac{k_{2}}{2}v^{2}, \tag{3.30}
\]
where \( v^{2} = y \) and \( k_{i}, \, i = 1, 2, \) are constants. (3.30) is solvable by means of elliptic functions. For \( n = -2, \) we have the following simplified equation and the canonical form of the equation:
\[
y_{0} = \text{arbitrary} : \, \, (r_{1}, r_{2}) = (0, 2), \tag{3.31}
y'' = \frac{4}{3} \frac{y'y''}{y} - \frac{3}{2} \left( \frac{y'}{y} \right)^{3},
\]
\[
y'' = \frac{4}{3} \frac{y'y''}{y} - \frac{3}{2} \left( \frac{y'}{y} \right)^{3} + k_{1} \frac{y'}{y}, \quad k_{1} = \text{constant.} \tag{3.32}
\]
respectively. Integrating (3.32) once gives
\[
y'' = \left( \frac{y'}{y} \right)^{2} + \frac{k_{2}}{2}y^{2} - \frac{k_{1}}{2}, \tag{3.33}
\]
where \( k_{2} \) is an integration constant. For \( n = 1, \) the simplified equation is
\[
y_{0} = \frac{6}{a} : \, \, (r_{1}, r_{2}) = (2, 6), \tag{3.34.a,b}
y'' = \frac{y'y''}{y} + ay'.
\]
Integration of (3.34.b) once yields
\[
y'' = ay^{2} + k_{1}y, \quad k_{1} = \text{constant.} \tag{3.35}
\]
(3.35) is solvable by means of elliptic functions. If \( A_{1} = A_{2} = 0, \) the canonical form of the equations is
\[
y'' = \frac{y'y''}{y} + 6yy' - \left( \frac{1}{24} \frac{k_{1}z^{2}}{z} + k_{2} + k_{3} \right) \frac{y'}{y} + k_{1}y
\]
+ \( \left( \frac{1}{12} \frac{k_{1}z^{2}}{z} + k_{2} \right), \tag{3.36}
\]
where \( k_{i}, \, i = 1, 2, 3, \) are constants. Integration of (3.36) once yields
\[
y'' = 6y^{2} + (k_{1}z + k_{4})y + \frac{1}{24} \frac{k_{1}z^{2}}{z} + k_{2} - k_{3}, \tag{3.37}
\]
\( k_{4} \) is constant.

If one lets \( y = v - (k_{1}z + k_{4})/12 \) in (3.37), then it yields the first Painlevé equation. If \( A_{1} = 1/2z, \, A_{2} = 0, \) then the canonical form of the equation is
\[
y'' = \frac{y'y''}{y} + 6yy' + \frac{1}{2z} \left( y'' - 6y^{2} \right) + \frac{5}{8} \frac{y}{z} \tag{3.38}
\]
\[
- \left( \frac{639}{5120} \frac{1}{z} - k_{1}z - k_{2} \right) \frac{y'}{y} - \left( \frac{5751}{1280} \frac{1}{z^{3}} \right) \left( \frac{k_{1}}{2z} + k_{1} \right). \tag{3.39.a,b}
\]
For \( n = 2, \) the simplified equation is
\[
y_{0} = \frac{4}{a} : \, \, (r_{1}, r_{2}) = (2, 4), \tag{3.39.a,b}
y'' = \frac{2}{3} \frac{y'y''}{y} - \left( \frac{y'}{y} \right)^{3} + ay'.
\]
Integration of (3.39.b) once yields
\[
y'' = \left( \frac{y'}{y} \right)^{2} - \frac{a}{2y^{2}} + k_{1}, \quad k_{1} = \text{constant.} \tag{3.40}
\]
(3.40) is of Painlevé type [14]. To obtain the canonical forms we have two possibilities, depending on the leading order \( \bar{a}. \) If \( A_{11} \neq 0 \) and \( A_{5} = A_{6} = A_{9} = A_{13} = 0, \) then \( \bar{a} = -1 \) and \( A_{11}(z_{0})u_{0}^{2} = 1, \)
Setting \(A_1 = 0\) and \(A_2 = 1/\zeta\), respectively, where \(k_1, k_2\) are constants. \(\alpha = -2\) is also a leading order, and the Fuchs indices are \((\tilde{r}_1, \tilde{r}_2) = (0, 2)\). The canonical equation is

\[
y'''' = 2\frac{y'''}{y} - \frac{(y')^3}{y^2} + 4yy' + k_1 \frac{y'}{y} + k_2 y^2 \frac{y'''}{y} - \frac{(y')^3}{y^2} + 4yy' - \frac{(y')^2}{y} - 4y^2 \]

\[- k_2 \frac{y}{\zeta} + k_1 \frac{y'}{y} + \frac{k_1}{2\zeta} \quad (3.41)\]

for \(A_1 = 0\) and \(A_2 = 1/\zeta\), respectively, where \(k_1, k_2\) are constants. \(\alpha = -2\) is also a leading order, and the Fuchs indices are \((\tilde{r}_1, \tilde{r}_2) = (0, 2)\). The canonical equation is

\[
y'''' = 2\frac{y'''}{y} - \frac{(y')^3}{y^2} + 4yy' + k_1 \frac{y'}{y} + k_2 y^2 \frac{y'''}{y} - \frac{(y')^3}{y^2} + 4yy' - \frac{(y')^2}{y} - 2y^2 + k_2 y. \quad (3.42)\]

For \(n = 4\), we have the following simplified equation

\[
y_0 = 3, \quad (r_1, r_2) = (2, 3), \quad (3.43.a,b)\]

\[
y'''' = \frac{5}{2} \frac{y'''}{y} - 3 \frac{(y')^3}{y^2} + ay'^2 \quad (4.3)\]

where \(a = 3\) and integrating (3.43.b) once yields

\[
y'' = \frac{(y')^2}{v} + v^3 + k_1, \quad k_1 = \text{constant} \quad (4.4)\]

where \(v^2 = y\). (3.44) is of Painlevé type [14]. If \(A_8 \neq 0\), then \(\alpha = -2\) and \(A_8 (c_0 - 1) = 1\), \((\tilde{r}_1, \tilde{r}_2) = (1, 2)\) and the canonical form is

\[
y'''' = \frac{5}{2} \frac{y'''}{y} - 3 \frac{(y')^3}{y^2} + 3y' + \frac{1}{6} A_8 \left[y'' - \frac{3}{2} \frac{(y')^2}{y} \right] \]

\[+ A_8 \frac{y'}{y} + \frac{4}{3} A_8 \quad (3.45)\]

where \(A_8\) satisfies

\[A_8 A_8' = \frac{2}{3} (A_8')^2 \quad (3.46)\]

If \(A_8 = k_1 = \text{constant}\), integration of (3.45) yields

\[
y'' = \frac{(y')^2}{v} + v^3 - 2k_1 \frac{1}{v} + k_2 \quad (3.47)\]

where \(v^2 = y\) and \(k_2\) is an integration constant. (3.47) is solvable by means of elliptic functions.

When \((c_1, c_2) = (3, -2)\), this case does not lead to any distinct Fuchs indices.

### 4. Leading Order \(\alpha = -3\)

\(\alpha = -3\) is also a possible leading order of equation (1.4). By adding the term \(y^2\), the following simplified equation with the leading order \(\alpha = -3\), is obtained:

\[
y'''' = c_1 \frac{y'''}{y} + c_2 \frac{(y')^3}{y^2} + ay^2, \quad (4.1)\]

where \(a\) is constant and \(c_1, c_2\) are given by (1.12), (1.15) and (1.16). In this case the Fuchs indices \(r\) and \(y_0\) satisfy the equations

\[
Q(r) = (r + 1)[r^2 - (13 - 3c_1) r + 60 - 36c_1 - 27c_2] = 0, \quad (4.2a)\]

\[
a y_0 = -60 + 36c_1 + 27c_2, \quad (4.2b)\]

respectively. (4.2.b) implies that there is only one branch. In order to have positive distinct Fuchs indices, \(3c_1\) and \(36c_1 + 27c_2\) both must be integers for all five cases.

To find the canonical forms of the equations, one should consider the following equations for \(c_2 = 0\) and \(c_2 \neq 0\)

\[
y'''' = c_1 y'' + ay^3 + A_1 y'' + A_2 (y')^2 + A_3 y y' \]

\[+ A_4 y^2 + A_5 y' + A_6 y + A_7 y + A_8, \quad (4.3)\]

\[
y'''' = c_1 y'' + c_2 (y')^3 + ay^3 + A_1 y'' + A_2 (y')^2 \]

\[+ A_3 y^2 y' + A_4 y + A_5 y'' + A_6 (y')^2 + A_7 y y' \]

\[+ A_8 y^2 + A_9 y' + A_{10} y + A_{11} y + A_{12}, \quad (4.4)\]

where the \(A_i\) are locally analytic functions of \(z\). The coefficients \(A_k\) can be found by using the same procedure described in the previous sections.

When \((c_1, c_2) = (3, -2 + 2/n^2)\), \(n\) takes the values of \(\pm 1, \pm 3\). But \(n = \pm 1\) does not lead a principal branch. For \(n = \pm 3\) we have the following simplified equation and the canonical form of the equation

\[
y_0 = \text{arbitrary} : \quad (r_1, r_2) = (0, 4), \quad (4.5)\]

\[
y'''' = \frac{3}{9} \frac{y'''}{y} - \frac{16}{9} \frac{(y')^3}{y^2}, \quad (4.6)\]

\[
y'''' = \frac{3}{9} \frac{y'''}{y} - \frac{16}{9} \frac{(y')^3}{y^2} + (k_1 z + k_2) y' + k_2 y, \quad (4.6)\]

\(k_1, k_2 = \text{constant}\).
When \((c_1, c_2) = (3 - 1/n, -2 + 1/n + 1/n^2)\), \(n\) takes the values of \(n = \pm 1, \pm 3\). There is no principal branch and there are no Fuchs indices for \(n = -1\) and \(n = 1\), respectively. Hence we have the following cases: For \(n = -3\), the simplified equation and the canonical equation are

\[
y_0 = \text{arbitrary} : (r_1, r_2) = (0, 3),
\]

\[
y''' = \frac{10}{3} y''' + \frac{20}{9} (y')^3 - \frac{20}{9} y^2,
\]  
(4.7)

respectively. For \(n = 3\), the simplified equation and the canonical equation are

\[
y_0 = \frac{6}{\alpha} : (r_1, r_2) = (2, 3),
\]

\[
y''' = \frac{8}{3} y''' - \frac{14}{9} (y')^3 + ay^2.
\]  
(4.9)

\[
y''' = \frac{8}{3} y''' - \frac{14}{9} (y')^3 + 6y^2 - ky^2 + k_1 y + \frac{k^2}{4},
\]  
(4.10)

respectively, where \(k_1 = \text{constant}\).

When \((c_1, c_2) = (3 - 3/n, -2 + 3/n - 1/n^2)\), \(n\) takes the values \(\pm 1, \pm 2, \pm 9\). But, we have the principal branch only for \(n = -3\), and the simplified equation and the canonical equation are:

\[
y_0 = \text{arbitrary} : (r_1, r_2) = (0, 1),
\]

\[
y''' = 4 y''' - \frac{28}{9} (y')^3,
\]  
(4.11)

\[
y''' = 4 y''' - \frac{28}{9} (y')^3 + Ay + A_3 y + A_4 y,
\]  
(4.12)

respectively, where \(A_3\) and \(A_4\) are arbitrary locally analytic functions of \(z\).

When \((c_1, c_2) = (3 - 2/n, -2 + 2/n)\), \(n\) takes the values of \(\pm 1, \pm 2, \pm 3\). But, we have only the following simplified equation and the canonical form which corresponds to \(n = 1\):

\[
y_0 = 1 : (r_1, r_2) = (4, 6), y''' = \frac{y'''}{y} - 24y^2,
\]  
(4.13)

\[
y''' = \frac{y'''}{y} - 24y^2 + k_1 y + \left(\frac{k^2}{12} z + k_2\right) y' - \frac{k^2}{12},
\]  
(4.14)

respectively, where \(k_1, k_2\) are constants.

When \((c_1, c_2) = (3, -2)\), this case does not lead any distinct Fuchs indices.

In conclusion, we obtained the canonical forms of non-polynomial third order equations with the leading orders \(\alpha = -1, -2, -3\), such that all pass the Painlevé test. Not the canonical forms, but the simplified equations, except (2.17), (2.26) and (2.94) given in section 2, were considered in the literature before [11, 12]. The simplified equations given in section 2 can be obtained by differentiating the leading terms of the third Painlevé equation and adding the terms of order \(-4\) as \(z \to z_0\) with constant coefficients, such that \(y = 0, \infty\) are the only singular values of the equation in \(y\), and they are of the order \(e^{-3}\) or greater, if one lets \(z = z_0 + \epsilon t\). Hence, these equations can be considered as the generalization of the third Painlevé equation.

In the third and fourth sections, we investigated the cases of leading order \(\alpha = -2, -3\), which were not considered before. We found that 20 new canonical forms of non-polynomial third order equations, namely (3.6), (3.9), (3.12), (3.15), (3.16), (3.19), (3.22), (3.26), (3.29), (3.32), (3.36), (3.38), (3.41), (3.42), (3.45), (4.6), (4.8), (4.10), (4.12), and (4.14), all pass the Painlevé test.

In the procedure we imposed the existence of at least one principal branch, i.e. the resonances are distinct positive integers for one branch. But the compatibility conditions at positive resonances for the second and third branches are identically satisfied for each case. Instead of having positive distinct integer resonances, one can consider the principal of distinct negative integer resonances. In this case it is possible to obtain equations belonging to Chazy classes.