Folded Solitary Waves and Foldons in the (2+1)-Dimensional Long Dispersive Wave Equation

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Z. Naturforsch. 58a, 280 – 284 (2003); received March 16, 2003

By means of the B"acklund transformation, a quite general variable separation solution of the (2+1)-dimensional long dispersive wave equation:

\begin{align*}
\lambda q_{tt} + q_{xx} - 2q \int (q^r) dy &= 0, \\
\lambda p_{tt} - p_{xx} + 2p \int (pq) dy &= 0,
\end{align*}

is derived. In addition to some types of the usual localized structures such as dromion, lumps, ring soliton and oscillated dromion, breathers soliton, fractal-dromion, peakon, compacton, fractal and chaotic soliton structures can be constructed by selecting the arbitrary single valued functions appropriately, a new class of localized coherent structures, that is the folded solitary waves and foldons, in this system are found by selecting appropriate multi-valued functions. These structures exhibit interesting novel features not found in one-dimensions. – PACS: 03.40.Kf., 02.30.Jr, 03.65.Ge.

Key words: Variable Separation Solution; the (2+1)-dimensional Long Dispersive Wave System; Folded Solitary Wave; Foldon.

\section{1. Introduction}

In the study of (2+1)-dimensional nonlinear physical systems, much effort has been focused on single valued localized excitations, such as solitons, dromions, rings, lumps, breathers, instantons, peakons, compactons, localized chaotic and fractal patterns, etc. [1]. However, in nature, there exist very complicated folded phenomena such as the folded protein [2], the folded brain and skin surface, and many many other kinds of folded biological systems [3]. The bubbles on (or under) a fluid surface may be thought to be the simplest folded waves. Further, various kinds of ocean waves are folded waves also. Loop solitons, which are thought as a class of simplest folded waves in the (1+1)-dimensional case, have been founded in many (1+1)-dimensional integrable systems [4] and have been applied in physical fields like the string interaction with an external field [5], quantum field theory [6] and particle physics [7]. Recently, Tang and Lou [8] first considered these special folded localized excitations and found folded solitary waves in some (2+1)-dimensional nonlinear models, such as the (2+1)-dimensional dispersive long wave equation, the (2+1)-dimensional Burgers equation, etc. They define a new type of soliton, called the foldon if the interaction between the folded solitary waves is completely elastic. In this paper, we further consider the (2+1)-dimensional long dispersive wave equation (LDWE)

\begin{align*}
\lambda q_{tt} + q_{xx} - 2q \int (pq) dy &= 0, \\
\lambda p_{tt} - p_{xx} + 2p \int (pq) dy &= 0,
\end{align*}

where $\partial_x = \partial_{\xi}$, $\partial_t = \partial_{\eta} - \lambda \partial_{\eta}$, and $\lambda$ is a constant. This system apparently differs from the usual or traditional long dispersive wave system [9], which has been introduced by Chakravarty, Kent, and Newman [10] by symmetrical reduction from the self-dual Yang-Mills field equation. It is interesting to note that (1) and (2) can be reduced to a single nonlocal equation introduced by Fokas [11],

\begin{equation}
 i\lambda q_{tt} + q_{xx} - 2q \int |q|^2 dy = 0,
\end{equation}

when $p = q^*$ and $t \rightarrow it$. Equation (3) arises in plasma physics under appropriate circumstances [12], and it admits exponentially localized solutions and satisfies
the Painlevé property [13]. Radha and Lakshmanan investigated the Painlevé integrability properties and bilinear forms [14] of (1) and (2). Velan and Lakshmanan proved that (1) and (2) admit an infinite-dimensional symmetry and presented some physically interesting solutions via invariance analysis [15]. Starting from a special Bäcklund transformation, we converted the LDWE into simple variable separation equations and obtained a quite general variable separation solution [16]. Since some types of the usual localized excitations of (1) and (2), such as multi-soliton, multidromion, multi-lump, multi-ring soliton and oscillating dromion solutions, fractal dromion, fractal lump, and chaotic dromion have been obtained by selecting some types of lower-dimensional appropriate functions, here we try to find some kinds of folded solitary waves and foldons for (1) and (2).

2. Variable Separation Solution for the (2+1)-dimensional LDWE

For the concreteness and completeness of the investigation we review first the variables separation procedure for (1) and (2).

Making use of the transformation

\[ pq = w_z, \]  

where \( w \) is some arbitrary potential, (1) and (2) can be converted into the partial differential equations

\[ \lambda q_x + q_{xx} - 2qw_x = 0, \]

\[ \lambda p_t - p_{xx} + 2pw_x = 0, \]

\[ pq = w_y, \]

where the integration constants in (5) and (6) have been set to zero. By taking the Bäcklund transformation

\[ q = \frac{Q}{f} + q_0, \quad p = \frac{P}{f} + p_0, \quad w = -(\ln f)_x + w_0, \]

which can be derived from the standard Painlevé truncated expansion, where \( f, P, Q \) are arbitrary differential functions of the arguments \( \{x, y, t\} \), and \((q_0, p_0, w_0)\) is an arbitrary seed solution, (5)–(7) become the following bilinear form

\[ (D_x^2 + \lambda D_t)Q \cdot f + q_0D_x^2 f \cdot f \]

\[ + f^2(q_{0xx} + \lambda q_{0x}) - 2fw_{0x}(Q + fq_0) = 0, \]

\[ (D_x^2 - \lambda P)P \cdot f + p_0D_x^2 f \cdot f \]

\[ + f^2(p_{0xx} - \lambda q_0) - 2fw_{0x}(P + fp_0) = 0, \]

\[ D_xD_yf \cdot f + 2(PQ + fPq_0) \]

\[ + fQp_0 + f^2p_0q_0 - f^2w_{0y} = 0, \]

where \( D_x, D_y, D_t \) are defined as

\[ D_x^nD_y^k f \cdot f = \lim_{x' = x, y' = y, t' = t} \left( \frac{\partial}{\partial x'} - \frac{\partial}{\partial y'} \right)^m \left( \frac{\partial}{\partial y'} - \frac{\partial}{\partial t'} \right)^k f(x', y', t', f(x', y', t') \]

which are the usual bilinear operators introduced first by Hirota [17].

To discuss further, we take the seed solution \((q_0, p_0, w_0)\) to be \( q_0 = 0, p_0 = 0, w_0 = F_0(x, t) \). Then (9)–(11) can be simplified to

\[ (D_x^2 + \lambda D_t)Q \cdot f - 2fQw_{0t} = 0, \]

\[ (D_x^2 - \lambda D_t)P \cdot f - 2fPw_{0t} = 0, \]

\[ D_xD_y f \cdot f + 2PQ = 0. \]

In order to find some interesting solutions of (13)–(15), we use the variable separation ansatz

\[ f = a_1 F + a_2 G + a_3 F G, \quad Q = F_1 G_1 \exp[\lambda(r + s)], \]

\[ P = F_1 G_1 / \exp[\lambda(r + s)], \]

where \( a_1, a_2, a_3 \) are arbitrary constants and \( F \equiv F(x, t), \ G \equiv G(y, t), \ F_1 \equiv F_1(x, t), \ G_1 \equiv G_1(y, t), \ r \equiv r(x, t), \ s \equiv s(y, t) \) are all arbitrary functions of the indicated variables.

Substituting the ansatz (16) into (15) yields

\[ F_1^2 G_1^2 - a_1 a_2 F_1 G_2 = 0. \]

Because the functions \( F \) and \( F_1 \) are only functions of \( \{x, t\} \) and the functions \( G \) and \( G_1 \) are only functions of \( \{y, t\} \), (17) can be solved by the following variable separated equations

\[ F_1 = \delta_1 \sqrt{a_1 a_2 c_0^{-1}} F_1, \quad G_1 = \delta_2 \sqrt{c_0 G_1}, \quad (\delta_1^2 = \delta_2^2 = 1). \]

(18)

Similar to the above procedure, substituting (16) with (17) into (13) and (14) yields the following variable separated equations

\[ -a_1 a_2 F_1 = 2a_1 a_2 x F_1 + c_1 F_1^2 + c_2 a_2 F + c_3 a_2^2, \]

(19)
where \( g(y), b(t) \) are arbitrary functions of the indicated variables.

Although it is still difficult to obtain general solutions of (19)–(22) for any fixed \( F_0 \), we can treat the problem in an alternative way. Because \( F_0 \) is an arbitrary seed solution, we can view \( F \) as an arbitrary function of \( \{x,t\} \). The function \( r \) can be expressed by \( F \) simply by integration from (19). Then the seed solution \( F_0 \) can be fixed by (21). The result reads

\[
\begin{align*}
    r_x &= -\frac{1}{2a_1a_2F_0}(c_1F^2 + c_2a_2F + c_3a_2^2 + a_1a_2F_1), \\
    F_{0t} &= \frac{1}{8F_x^2}[2F_xF_{xxx} + 4\lambda^2F_x^2(r_x^2 + r_x + b_1) - F_{xx}^2],
\end{align*}
\]

As to the Riccati equation (20), its general solution has the form

\[
G(y,t) = \frac{A_1(t)}{A_2(t) + U(y)} + A_3(t),
\]

where \( U(y) \) is an arbitrary function of \( y \) while \( A_1, A_2, \) and \( A_3 \) are arbitrary functions of \( t \), which are linked with \( c_1, c_2, \) and \( c_3 \) by

\[
\begin{align*}
    c_1 &= -\frac{1}{A_1}(2a_1a_2A_3A_{2t} + a_1a_3A_{1t} + a_2^2A_2), \\
    c_2 &= -\frac{1}{A_1}(2a_1A_3A_{2t} + a_1A_{1t}) \\
    c_3 &= -\frac{1}{A_1}(A_3A_{1t} + A_2^2A_{2t} - A_1A_{3t}).
\end{align*}
\]

Using the relations (26)–(28), (20) becomes

\[
\begin{align*}
    G_t &= -\frac{1}{A_1}[A_2^2G^2 - (A_{1t} + 2A_2A_2)G] \\
    &= + A_2^2A_2 + A_3A_{1t} - A_1A_{3t}.
\end{align*}
\]

One can verify directly that (25) is a general solution of (29).

Finally, substituting (16) with (18)–(22) into (9), we derive a quite general solution of the (2+1)-dimensional system (5)–(7):

\[
\begin{align*}
    q &= \frac{\delta_1 \delta_2 \sqrt{a_1a_2F_0G} \exp[\lambda(r + g(y) + b(t))]}{a_1F + a_2G + a_3FG}, \\
    p &= \frac{\delta_1 \delta_2 \sqrt{a_1a_2F_0G}}{(a_1F + a_2G + a_3FG) \exp[\lambda(r + g(y) + b(t))]}, \\
    w &= F_0 - \frac{a_1F + a_3FG}{a_1F + a_2G + a_3FG},
\end{align*}
\]

with the four arbitrary functions \( F(x,t), G(y,t), g(y), \) and \( b(t) \), while \( F_0 \) and \( r \) are determined by (23) and (24). In the following discussion we study the structure of the potential \( qp \). From the solutions (30) and (31) we have

\[
qp = \frac{a_1a_2F_0G}{(a_1F + a_2G + a_3FG)^2}.
\]

### 3. Folded Solitary Waves and Foldons of the (2+1)-dimensional LDWE

It is of interest to mention that the solution (33) is valid for many (2+1)-dimensional models, and that many types of solutions can be obtained due to the arbitrariness of the functions \( F \) and \( G \) included in (33). Now we study the folded solitary waves and foldons. Starting from the the expression (33), these special solutions should be described by multi-valued functions. We write a localized function \( \varphi(x,t) \) in the form

\[
\varphi(x,t) = \sum_{j=1}^{M} f_j(\xi + v_xt) + \sum_{i=1}^{N} g_j(\xi + v_xt),
\]

where \( v_1 < v_2 < \ldots < v_M \) are all arbitrary constants and \((f_j, g_j)\) and \(v_j\) are all localized functions with \( f_j(\pm \infty) \) and \( g_j(\pm \infty) = G^\pm \) being constants. From the second equation of (34) we know that \( \xi \) may be a multi-valued function in some possible regions of \( x \) by selecting the function \( g_j \) suitably. So the function \( \varphi(x,t) \) may be a multi-valued function of \( x \) in these regions, though it is a single valued function of \( \xi \). It is also clear that \( \varphi(x,t) \)
Concrete choices are: (a) $F_x = \text{sech}^2(\xi - v_1 t)$, $G_y = \text{sech}^2 \theta$, $x = \xi - 1.2 \tanh(\xi - v_1 t), y = \theta + k_0 \tanh \theta, F = \int_0^\xi F_x \text{sech}^2(\xi - v_1 t) + l_0, G = \int_0^\theta G_{y \theta} d\theta + h_0$ and $a_1 = 1, a_2 = -2, a_3 = 0, k_0 = 1, l_0 = 8, h_0 = 0$; (b) same as (a) but with $k_0 = -1.1$; (c) $F_x, F, G$ and $a_1, a_2, a_3$ are the same as (a), however $G_y = \text{sech}^2 + \text{sech}^6 \theta, x = \xi + 2 \tanh(\xi - v_1 t) + \tanh^2(\xi - v_1 t) - k_0 \tanh^3(\xi - v_1 t), y = \theta + 2 \tanh \theta + \tanh^2(\theta) - k_0 \tanh^3(\theta)$ and $l_0 = 50, h_0 = 50, k_0 = 5.5$; (d) same as (c) but with $k_0 = 250, k_0 = 10$.}

is an interaction solution of $M$ localized solutions because of the property $\xi \big|_{x \to \infty} \to \infty$. Now, if we specify different functions appearing in the formula (33), then we can get various types of folded solitary waves and foldons.

In Fig. 1, four typical special folded solitary waves are plotted for the field quantity $pq$ defined by (33) with $F_x = 0.8 \text{sech}^2(\xi) + 0.5 \text{sech}^2(\xi - 0.25 t)$, $G_y = \text{sech}^2 \theta, x = \xi - 1.5 \tanh(\xi) - 1.5 \tanh(\xi - 0.25 t), y = \theta - 2 \tanh \theta, F = \int_0^\xi F_x \text{sech}^2(\xi - v_1 t) + 20, G = \int_0^\theta G_{y \theta} d\theta$ and $a_1 = 1, a_2 = -2, a_3 = 0$. (a) $t = -18$, (b) $t = 7.2$, (c) $t = 18$. (d) $t = -18$.

$G_y = \sum_{j=1}^M Q_j(\theta), y = \theta + R(\theta), G = \int_0^\theta G_{y \theta} d\theta + h_0.\quad (35)$

where $l_0$ and $h_0$ are arbitrary integration constants. In (35), $Q_j(\theta), \forall j$, and $R(\theta)$ are localized functions of $\theta$. The more detailed choices of functions in the figures are directly given in the figure legends.
Figures 2a–c show the possible existence of foldons which are given by (33), and the concrete choice of the functions is given in the figure legends. From Fig. 2a and 2c we can see that the interaction of two foldons is completely elastic. Because one of the velocities of foldons has been chosen as zero, it can also be seen that there are phase shifts for the two foldons. Especially, before the interaction, the static foldon (the large one) is located at \( x = -1.5 \), and after the interaction the large foldon is shifted to \( x = 1.5 \).

4. Summary and Discussion

In summary, with the help of the Bäcklund transformation, a rather general variable separation solution for the (2+1)-dimensional LDWE is found. Starting from the formula for the quantity \( pq \) expressed by (33) of the (2+1)-dimensional LDWE, folded solitary waves and foldons have been obtained by choosing arbitrary functions appropriately. We can see that the foldons may be folded quite arbitrarily and complicatedly, and that they possess quite rich structures and interaction behaviors. Explicit phase shifts for the localized structures defined by (33) have been given. In the natural world there exist many complicated folded and/or multi-valued phenomena. Seeking folded solitary waves and foldons and solutions of their interaction in other high dimensional nonlinear models seems useful and should be studied further.

Acknowledgement

One of the authors (Zhang Jiefang) is in debt to thank for the useful discussions with Professor Lou Senyue. We also thank Professor Roger Grimshaw and Doctor Rod Halburd in Loughborough University for their help. One of the referee’s careful correction of the manuscript is also appreciated. This work is supported by the Pao Yu-Kong and Pao Zhao-long Scholarship for Chinese Students Studying Abroad, the Foundation of “151 Talent Engineering” of Zhejiang Province and the Natural Science Foundation of China (Grant No. 10272072).