On Diffusion in Some Biological and Economic Systems

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It is argued that diffusion in biological and economical systems is better modelled by Cattaneo's equation where memory effects are included. Reaction diffusion equations using Cattaneo's system are derived for prisoner's dilemma (PD) and hawk-dove (HD) games. Nonlinear wave solutions are derived for them. As expected the asymptotic solution for the PD case is insufficient. Hence a cellular automata motivated by Cattaneo's system is used to show the existence of cooperation in the case of local game.

Key words: Cattaneo System; Cellular Automata; Non-linear Waves.

1. Cattaneo's Diffusion

The standard diffusion equation depends on the continuity equation and Fick's law

$$j(x,t) = -D\partial c(x,t)/\partial x, \qquad (1)$$

where j is the diffusing object (e. g. technology, concept, etc...), c is the distribution function of this object and D is the diffusion constant. The resulting standard diffusion equation is

$$\partial c(x,t)/\partial t = D\partial^2 c(x,t)/\partial x^2. \qquad (2)$$

A basic weakness of this equation is that the flux j reacts simultaneously to the gradient of c and consequently an unbounded propagation speed is assumed. This manifests itself in many solutions to (2) e. g. $c(x,t) = 1/\sqrt{4\pi Dt}\exp(-x^2/4Dt)$, $c(x,0) = \delta(x)$, i. e. $c(x,t) > 0\,\forall x$, $\forall t > 0$. This is unrealistic specially in biological and economical systems where it is known that propagation speeds are typically small. To rectify this weakness Cattaneo [1] replaced Fick's law (1) by

$$j + \tau\partial j/\partial t = D\partial c/\partial x. \qquad (3)$$

The resulting Cattaneo's diffusion equation is

$$\tau\partial^2 c/\partial t^2 + \partial c/\partial t = D\partial^2 c/\partial x^2, \qquad (4)$$

where τ is a time constant which measures the memory effect as will be shown. The corresponding Cattaneo's reaction diffusion is [2]

$$\tau\partial^2 c/\partial t^2 + (1-\tau df/dc)\partial c/\partial t = D\nabla^2 c + f(c). \qquad (5)$$

The time constant τ can be related to the memory effect of the flux j as a function of the distribution c as follows [3]: Assume

$$j(x,t) = -\int_0^t K(t-t')\partial c(x,t')/\partial x \,\mathrm{d}t', \qquad (6)$$

hence $j + \tau\partial j/\partial t = -\tau K(0)c(x,t) - \int_0^t(\tau\partial K(t-t')/\partial t + K(t-t'))\partial c/\partial x\,\mathrm{d}t'$. This equation is equivalent to Cattaneo's (3) if $K(t) = D/\tau\exp(-t/\tau)$.

This further supports that Cattaneo's diffusion is more suitable for economic and biological systems than the usual one since, e. g., it is known that we take our decisions according to our previous experiences so memory effects are quite relevant.

2. Cattaneo's Reaction-Diffusion for Hawk-Dove Game

Hawk-dove (HD) game is an interesting one both mathematically and biologically [4, 5]. It has two strategies H, D and the payoff matrix is

$\begin{bmatrix} & H & D \\ H & 1/2(v-c) & v \\ D & 0 & v/2 \end{bmatrix}$. This means that (for two similar players) every player can play hawk (H) or dove (D). if both play hawk then each of them gets a reward $(v-c)/2$. If one plays hawk and the other plays doth then the first gets v while the other gets zero. If both play dove then each gets $v/2$ For $0 < c < v$ the solution is to adopt H. But if $0 < v < c$ the max-min solution of von-Newman et al is to adopt D. However It has been shown [5] that this solution is unstable since a mutant adopting H in a population of doves will win so much that it will encourage others to mutate. This will continue till the number of hawks is large enough for fights between them becoming frequent. Hence they start losing (payoff = $1/2(v-c) < 0$). Then H is no longer adopted. At steady state the fraction p of hawks is obtained by equating the payoffs of both strategies and one gets $p = v/c$.

Considering the spatial effects, Cattaneo's reaction-diffusion equation (5) for the HD game is

$$\tau\partial^2 p/\partial t^2 + (1 - \tau \mathrm{d}f/\mathrm{d}p)\partial p/\partial t = D\nabla^2 p + f(p), \quad f(p) = -p(1-p)(pc - v)/2. \tag{7}$$

In general, for 2×2 symmetric games with payoff matrix $[a_{ij}]$ $f(p)$ is given by

$$f(p) = p(1-p)[(a_{12} - a_{22}) + p(a_{11} + a_{22} - a_{12} - a_{21})].$$

We look for a wave solution of the form $p = p(\zeta)$, $\zeta = x - c_0 t$. The function $f(p)$ satisfies $f(0) = f(1) = f(v/c) = 0$, $f'(0) > 0$, $f'(1) > 0$, $f'(v/c) < 0$, where $f' = \mathrm{d}f/\mathrm{d}\zeta$. There are two expected waves in this system. The first begins at $p = 0$ and ends at $p = v/c$ while the second begins at $p = 1$ and ends at $p = v/c$. Equation (7) now becomes

$$\tau c_0^2 p'' - c_0[1 - \tau \mathrm{d}f/\mathrm{d}p]p' = Dp + f(p). \tag{8}$$

Using the standard analysis of nonlinear waves [2] one gets:

Proposition 1. a) The necessary conditions for the existence of a wave solution for (8) starting at $p = 0$ and ending at $p = v/c$ are

$$\tau v < 2, \text{ and } c_0 \geq \sqrt{2vD/[(1-\tau v)^2 + 2\tau v]}. \tag{9}$$

b) The necessary conditions for the existence of a wave solution for (8) starting at $p = 1$ and ending at $p = v/c$ are

$$\tau(c - v) < 2, \text{ and} \tag{10}$$

$$c_0 \geq \sqrt{2(c-v)D/[\{1 - \tau(c-v)\}^2 + 2\tau(c-v)]}.$$

An interesting choice is $c_0 = \sqrt{D/\tau}$ which reduces (8) to first order and the solution is given implicitly by

$$p^\alpha (1-p)^\beta (p - v/c)^{-\gamma} = A\exp(-\sqrt{\tau/D}\zeta), \tag{11}$$

$$\alpha = 2/v - \tau, \ \beta = 2/(c-v) - \tau,$$

$$\gamma = 2/[v(1 - v/c)] + \tau,$$

and A is a constant to be determined by the initial conditions.

It is clear that this solution does not exist for the standard reaction diffusion since $\tau = 0$ in this case. Also it is clear that the nonlinear wave solutions (9), (10), (11) give the correct asymptotic solution $p = v/c$.

3. Cattaneo's Reaction-Diffusion for Prisoner's Dilemma Game

Prisoner's dilemma (PD) is a 2×2 symmetric game in which two possible strategies cooperate (C) or defect (D). The payoff matrix is $\begin{bmatrix} & C & D \\ C & R & S \\ D & T & U \end{bmatrix}$, where $T > R > U > S$ and $2R > T + S$. Define $\gamma = U - S$ and $\alpha = R + U - S - T$. Then $\gamma > 0$ while α can be positive, negative or zero. Then $f(p)$ for PD is

$$f(p) = -p(1-p)(\gamma - \alpha p). \tag{12}$$

We look for nonlinear wave solutions, hence it is assumed that $p = p(\zeta)$, $\zeta = x - c_0 t$. Choosing $c_0 = \sqrt{D/\tau}$ one gets the implicit solution

$$p^{(\tau+1/\gamma)}(1-p)^{[\tau+1/(\alpha-\gamma)]}|p - \gamma/\alpha|^{[\tau+\alpha/\{\gamma(\alpha-\gamma)\}]} = A\exp(\zeta\sqrt{\tau/D}), \tag{13}$$

where A is a constant to be determined from initial conditions. A wave starting at $p \simeq 1$ (almost all cooperate) as $t \to -\infty\,(\zeta \to \infty)$ and ending at $p = 0$ as $t \to \infty$ is represented by (13) provided that $\tau < 1/(\gamma - \alpha)$. Notice that if $\alpha > 0$ then $\gamma/\alpha > 1$ hence $p < \gamma/\alpha$.

For the case $\alpha = 0$ the choice $c_0 = \sqrt{D/\tau}$ gives

$$p^{(\tau+1/\gamma)}(1-p)^{(\tau-1/\gamma)} = A\exp\left(\zeta\sqrt{\tau/D}\right) \quad (14)$$

Again this solution represents a wave starting from $p \simeq 1$ and ending at $p = 0$ provided that $\tau < 1/\gamma$.

For the standard reaction diffusion case ($\tau = 0, \gamma > 0 > \alpha$) an exact solution can be obtained in the form (here we set $D = 1$ for simplicity)

$$p(x,t) = 1/\{1+\exp[-x\sqrt{|\alpha|/2}+t(\gamma-\alpha/2)]\}. \quad (15)$$

As t increases, $p \to 0$.

All the previous solutions agreed on the asymptotic solution $p = 0$ i. e. all defect which is unrealistic. It has been shown [6] that in local PD games cooperation exists, hence in the next section we will use a cellular automata (CA) motivated by Cattaneo's idea [7] to model local PD games.

4. Cattaneo CA

CA [8] is a quite useful tool in studying dynamic and/or nonequilibrium [9], and/or spatially inhomogeneous [10, 11] systems. In this sense it complements differential equations. A standard CA in nonequilibrium statistical mechanics is the Domany-Kinzel (DK) model [12] which is a probabilistic CA model with the following rules: Consider a $1-d$ ring with N sites. Let $x_i(t) \in \{0,1\}$ be the state at site i at time t and let sum = $x_i(t)+x_{i+1}(t)$. Then the updating rules are

1) If sum = 0 then $x_i(t+1) = 0$.

2) If sum = 1 then $x_i(t+1) = 1$, with probability p_1.

3) If sum = 2 then $x_i(t+1) = 1$, with probability p_2.

Recently Bagnoli et al. [13] have proposed another CA with the following rules: Let sum = $x_i(t)+x_{i+1}(t)+x_{i-1}(t)$. Then

1) If sum = 0 then $x_i(t+1) = 0$.

2) If sum = 1 then $x_i(t+1) = 1$, with probability p_1.

3) If sum = 2 then $x_i(t+1) = 1$, with probability p_2.

4) If sum = 3 then $x_i(t+1) = 1$.

Both these models correspond to the stochastic reaction-diffusion equation

$$\partial c/\partial t = D\nabla^2 c + f(c) + \sqrt{c}\epsilon, \quad (16)$$

where c is the concentration of live ($x_i(t) = 1$) sites, ϵ is a zero-mean Gaussian random variable with unit variance. As we argued before constructing a CA corresponding to Cattaneo's system is an important problem. Discretizing (7) the following CA (we call it Cattaneo CA) is proposed: Define sum1 = $x_i(t)+x_{i+1}(t)$, sum2 = $x_i(t+1)+x_{i+1}(t)$, $\omega \in [0,1]$,

$$\text{sum} = \text{int}[\omega * \text{sum1} + (1-\omega) * \text{sum2} + .5], \quad (17)$$

where int$[x]$ is the integer part of x. Then

1) If sum = 0 then $x_i(t+2) = 0$.

2) If sum = 1 then $x_i(t+2) = 1$ with probability p_1.

3) If sum = 2 then $x_i(t+2) = 1$ with probability p_2.

The corresponding mean field dynamical system is

$$x(t+1) = y(t), \quad (18)$$

$$y(t+1) = \omega\{p_1[y(1-x)+x(1-y)]+p_2yx\} + (1-\omega)[2p_1x(1-x)+p_2x^2]$$

The steady states are $x = 0, (2p_1-1)/(2p_1-p_2)$. The second solution exists only if $p_1 > 0.5$. Stability analysis shows that $c = 0$ is stable if $p_1 < 0.5$ while the nonzero solution is stable if $p_1 > .5$ therefore the transition between the two phases is expected to be of second order.

Now we explain the CA in terms of PD game. Identify $x_i = 1$ (0) with a cooperator (defector), respectively. It is resonable to assume that a player will cooperate with probability p_1 if one of his neighbors cooperates. The case when both neighbors cooperate is interesting because if the player i is "nice" he will cooperate as well i. e. p_2 is large. If he is "exploiting" then p_2 is very small. In our simulations we counted the denisty of cooperators after a long time of play and we found that for the case of "nice" players ($p_2 \simeq 1$) cooperation exists for $p_1 \geq 0.5$ while for "exploiters" cooperation persists if $p_1 \geq 0.8$ for different values of ω. In our numerical simulations $N = 500, T = 10000$.

[1] G. Cattaneo, Atti. Sem. Mat. Fis. Univ. Modena **3,** 83 (1948).
[2] K. P. Hadeler, in O. Diekman et al. (eds), Mathematics inspired by Biology, Springer, Berlin 1998.
[3] A. Compte and R. Metzler, J. Phys. A **30,** 7277 (1997).
[4] J. Hofbauer and K. Sigmund, Evolutionary Games and Population Dynamics, Cambridge U. P., Cambridge 1998.
[5] J. Maynard Smith, Evolution and the Theory of Games, Cambridge U. P., Cambridge 1982.
[6] M. A. Nowak and R. M. May, Int. J. Bifur. and Chaos **3,** 35 (1993).
[7] E. Ahmed and A. S. Elgazzar, in preparation.
[8] F. Bagnoli, Cond-mat 9810012 (1998).
[9] P. Grassberger, J. Stat. Phys. **79,** 13 (1995).
[10] K. Ito and M. Matsuzaki, J. Geophys. Res. B **45,** 6853 (1990).
[11] E. Ahmed and H. N. Agiza, Physica A **253,** 347 (1998).
[12] E. Domany and W. Kinzel, Phys. Rev. Lett. **53,** 447 (1994).
[13] F. Bagnoli, N. Boccara, and R. Rechtman, Cond-mat 0002361 (2000).