(3+1)-Dimensional Integrable Models with Infinitely Dimensional Virasoro Type Symmetry Algebra and the Painlevé Property

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In this paper, some Virasoro integrable models are obtained by means of the realizations of the generalized centerless Virasoro-type symmetry algebra, $[\sigma(f_1), \sigma(f_2)] = \sigma(\dot{f}_2 f_1 - \dot{f}_1 f_2)$. It is interesting that some of them may be not only Virasoro integrable but also Painlevé integrable.

Key words: Virasoro Symmetry Algebra; (3+1)-Dimensional Integrable Model; Group Invariant Equation; Painlevé Integrability.

1. Introduction

The soliton theory has attracted much attention from both physicists and mathematicians because it has been widely applied in many physically significant fields (such as fluids, optics and astrophysics, etc.) [1]. (1+1)- and (2+1)-dimensional integrable models have been deeply investigated. However, there is little progress in the study of $(n + 1)$-dimensional ($n \geq 3$) integrable models though many physicists and mathematicians have tried to find some significant (3 + 1)-dimensional integrable models [2, 3].

Recently, one of the present authors has proposed some possible methods to search for some nontrivial higher-dimensional integrable models under some special meanings [4, 5, 6]. For instance, basing on the fact that all the known (2+1)-dimensional integrable models possess a common generalized centreless Virasoro type symmetry algebra,

$$[\sigma(f_1), \sigma(f_2)] = \sigma(\dot{f}_2 f_1 - \dot{f}_1 f_2), \qquad (1)$$

while all the known (2+1)-dimensional nonintegrable models do not possess this type of symmetry algebra, we had defined [5, 7] a special type of integrability under the meaning that a model (or its variant form) possesses an infinite dimensional centreless Virasoro type symmetry algebra. For convienience later, we call this types of integrability as the Virasoro integrability. In (1), f_1 and f_2 are *arbitrary functions* of a single independent variable, say time t, and the dots over the functions f_1 and f_2 present the derivatives with respect to the independent variables.

It is known that, when we say a model is integrable, we should point out that the model is integrable under what meaning? We may say a model is Painlevé integrable if the model possesses the Painlevé property, it is IST integrable if the model can be solved by the inverse scattering transformation, Lax integrable if the model possesses a Lax pair, etc. Now it is a natural and important question if can we find some higher dimensional models which are not only Virasoro integrable but also Painlevé integrable, IST integrable or Lax integrable? In this paper we try to find some (3+1)-dimensional models which have the Virasoro integrability and the Painlevé integrability at the same time via some concrete realizations of the Virasoro-type symmetry algebra (1).

In Sect. 2, we sketch the general method to realize the Virasoro symmetry algebra (1). In Sect. 3, a concrete realization of the Virasoro symmetry algebra (1) is used to construct (3+1)-dimensional Virasoro integrable models. In Sect. 4 we check the Painlevé integrability for the Virasoro integrable models obtained in the Sect. 3 via the Weiss-Tabor-Carnevale-Kruskal (WTCK) approach. Section 5 a short summary and discussion is given.

2. General Theory

In order to find out the invariant equations of symmetry algebra (1), we have to realize the Lie algebra (1) in terms of vector fields on the space $S\otimes U$ of independent and dependent variables. In our case, S is the four-dimensional space-time with coordinates (x, y, z, t) and U is the space of real scalar functions $u(x, y, z, t)$. For Lie point symmetries, the vector field V of symmetries in the space $S\otimes U$ have the generalized form

$$V = X(x, y, z, t, u)\partial_x + Y(x, y, t, u)\partial_y \tag{2}$$
$$+ Z(x, y, t, u)\partial_z + T(x, y, t, u)\partial_t + U(x, y, t, u)\partial_u.$$

To realize the algebra (1), we can select f in (1) as an arbitrary function of t and restrict T, X, Y, Z and U as follows

$$T = f(t), \tag{3}$$
$$\{X, Y, Z, U\} = \Big\{ \sum_{i=1}^{n} f^{(i)}X_i, \sum_{i=1}^{n} f^{(i)}Y_i, \sum_{i=1}^{n} f^{(i)}Z_i, \sum_{i=1}^{n} f^{(i)}U_i\Big\},\ n = 1, 2, 3, ...,$$

where $f^{(i)} = \mathrm{d}^i f/\mathrm{d}t^i$ and X_i, Y_i, Z_i and U_i are functions of x, y, z, t, u and should be selected to satisfy the commutation relation (1). In order to construct invariant k^{th}-order PDEs, we have to know how the considered group acts on the first k^{th}-order derivatives $u_x,\ u_y,\ u_z,\ u_t, ..., u_{x^iy^jz^mt^r} \equiv \partial_x^i\partial_y^j\partial_z^m\partial_t^r u, (1 \le i+j+m+r \le k)$ once we know its action on (x, y, t, u). Since our entire approach is infinitesimal, it is sufficient for us to know the k^{th} prolongation of the vector field $\boldsymbol{V}$. The general formula for the k^{th} prolongation of a vector field $\boldsymbol{V}$ is given by [8]

$$\mathrm{pr}^k V = V + U^x\partial_{u_x} + U^y\partial_{u_y} + U^z\partial_{u_z} + U^t\partial_{u_t} + \ldots + \sum_{1\le i+j+m+n\le k} U^{x^iy^jz^mt^n}\partial_{u_{x^iy^jz^mt^n}}, \tag{4}$$

$$U^x = D_x(U - Xu_x - Yu_y - Zu_z - Tu_t) + Xu_{xx} + Yu_{xy} + Zu_{xz} + Tu_{xt}, \tag{5}$$

$$U^y = D_y(U - Xu_x - Yu_y - Zu_z - Tu_t) + Xu_{xy} + Yu_{yy} + Zu_{yz} + Tu_{yt}, \tag{6}$$

$$U^z = D_z(U - Xu_x - Yu_y - Zu_z - Tu_t) + Xu_{xz} + Yu_{yz} + Zu_{zz} + Tu_{zt}, \tag{7}$$

$$U^t = D_t(U - Xu_x - Yu_y - Zu_z - Tu_t) + Xu_{xt} + Yu_{yt} + Zu_{zt} + Tu_{tt}, \tag{8}$$

$$U^{x^iy^jz^mt^n} = D_xU^{x^{i-1}y^jz^mt^n} - (D_xX)u_{x^iy^jz^mt^n} - (D_xY)u_{x^{i-1}y^{j+1}z^mt^n} - (D_xZ)u_{x^{i-1}y^jz^{m+1}t^n} - (D_xT)u_{x^{i-1}y^jz^mt^{n+1}}, \tag{9}$$

$$U^{x^iy^jz^mt^n} = D_yU^{x^iy^{j-1}z^mt^n} - (D_yX)u_{x^{i+1}y^{j-1}z^mt^n} - (D_yY)u_{x^iy^jz^mt^n} - (D_yZ)u_{x^iy^{j-1}z^{m+1}t^n} - (D_xT)u_{x^iy^{j-1}z^mt^{n+1}}, \tag{10}$$

$$U^{x^iy^jz^mt^n} = D_zU^{x^iy^jz^{m-1}t^n} - (D_zX)u_{x^{i+1}y^jz^{m-1}t^n} - (D_zY)u_{x^iy^{j+1}z^{m-1}t^n} - (D_zZ)u_{x^iy^jz^mt^n} - (D_zT)u_{x^iy^jz^{m-1}t^{n+1}}, \tag{11}$$

$$U^{x^iy^jz^mt^n} = D_tU^{x^iy^jz^mt^{n-1}} - (D_tX)u_{x^{i+1}y^jz^mt^{n-1}} - (D_tY)u_{x^iy^{j+1}z^mt^{n-1}} - (D_tZ)u_{x^iy^jz^{m+1}t^{n-1}} - (D_tT)u_{x^iy^jz^mt^n}, \tag{12}$$

where D_x, D_y, D_z and D_t are total derivatives. In order to obtain some explicit invariant equations, we can choose a concrete realization $\sigma = V_1$ which satisfies the Virasoro type algebra (1) and calculate the kth prolongation. We know that the generalized V_1 invariant equations should have the form [9]

$$\Delta(x, y, z, t, u, u_x, u_y, u_z, u_t, ..., u_{x^iy^jz^mt^n}, ...) = 0, \tag{13}$$

where the function Δ satisfies

$$\mathrm{pr}^{(k)}V_1 \cdot \Delta = 0 \tag{14}$$

To find such types of group invariant equations, we should solve the corresponding characteristic equations of (4) in which all the arguments in (3) are viewed as independent variables. Solving the characteristic equations, we can get a set of elementary invariants, $I_r(x, y, z, t, u, ..., u_{x^iy^jz^pt^q})$,

($1 \leq i+j+p+q \leq k$, $r = 1, 2, 3 \ldots$). The general V_1 invariant equation then can be written as

$$H(I_1, I_2, I_3, \ldots, I_r, \ldots) = 0. \tag{15}$$

Usually, the group invariants, I_r, are f dependent. However in the definition of the Virasoro integrability, the model should be f independent. So, to find the Virasoro integrable models, we should select the f-independent models from (15).

3. (3+1)-Dimensional Virasoro Integrable Models

In order to get some concrete invariant equations, we have to find some concrete realizations of the Virasoro-type symmetry algebra (1). In this paper we fix the realization as

$$\sigma_1 = V_1 = f(t)\partial_t + (c_2 x\dot{f} + c_5\ddot{f})\partial_x + c_3 y\dot{f}\partial_y + c_4 z\dot{f}\partial_z + (c_1 u\dot{f} + c_6 xyz\ddot{f})\partial_u, \tag{16}$$

where c_1, c_2, c_3 and c_4 satisfy $c_2 + c_4 + c_3 - c_1 = 1$, $c_2 = -1$ (if $c_5 \neq 0$). We can easily prove that $\sigma_1 = V_1$ satisfies (1). Using the formulas (5) - (12), the corresponding k^{th} prolongation of the vector field (16) is

$$\begin{aligned}
\mathrm{pr}^{(k)}V_1 = {} & V_1 + [(c_1 - c_2)\dot{f}u_x + c_6 yz\ddot{f}]\partial_{u_x} + [(c_1 - c_3)\dot{f}u_y + c_6 xz\ddot{f}]\partial_{u_y} + [(c_1 - c_4)\dot{f}u_z + c_6 xy\ddot{f}]\partial_{u_z} \\
& + [(c_1 - 1)\dot{f}u_t + (c_1 u - c_2 x u_x - c_3 y u_y - c_4 z u_z)\ddot{f} + (-c_5 u_x + c_6 xyz)f^{(3)}]\partial_{u_t} \\
& + [(c_1 - c_2 - 1)\dot{f}u_{xt} + ((c_1 - c_2)u_x - c_2 x u_{xx} - c_3 y u_{xy} - c_4 z u_{xz})\ddot{f} + (c_6 yz - c_5 u_{xx})f^{(3)}]\partial_{u_{xt}} \\
& + [(c_1 - c_3 - 1)\dot{f}u_{yt} + ((c_1 - c_3)u_y - c_2 x u_{xy} - c_3 y u_{yy} - c_4 z u_{yz})\ddot{f} + (c_6 xz - c_5 u_{xy})f^{(3)}]\partial_{u_{yt}} \\
& + [(c_1 - c_4 - 1)\dot{f}u_{zt} + ((c_1 - c_4)u_z - c_2 x u_{xz} - c_3 y u_{yz} - c_4 z u_{zz})\ddot{f} + (c_6 xy - c_5 u_{xz}f^{(3)}]\partial_{u_{zt}} \\
& + [(c_1 - c_2 - c_3)\dot{f}u_{xy} + c_6 z\ddot{f}]\partial_{u_{xy}} + [(c_1 - c_2 - c_4)\dot{f}u_{xz} + c_6 y\ddot{f}]\partial_{u_{xz}} \\
& + [(c_1 - c_3 - c_4)\dot{f}u_{yz} + c_6 x\ddot{f}]\partial_{u_{yz}} + \sum_{n=2}^{k}(c_1 - nc_2)\dot{f}u_{nx}\partial_{u_{nx}} + \sum_{n=2}^{k}(c_1 - nc_3)\dot{f}u_{ny}\partial_{u_{ny}} \\
& + \sum_{n=2}^{k}(c_1 - nc_4)\dot{f}u_{nz}\partial_{u_{nz}} + [(c_1 - c_2 - c_3 - c_4)\dot{f}u_{xyz} + c_6\ddot{f}]\partial_{u_{xyz}} \\
& + \sum_{3 \leq n+m+r \leq k}(c_1 - nc_2 - mc_3 - rc_4)\dot{f}u_{x^n y^m z^r}\partial_{u_{x^n y^m z^r}} \\
& + [(c_1 - 2c_2 - 1)\dot{f}u_{xxt} - ((2c_2 - c_1)u_{xx} + c_2 x u_{xxx} + c_3 y u_{xxy} + c_4 z u_{xxz})\ddot{f} - c_5 u_{xxx}f^{(3)}]\partial_{u_{xxt}} \\
& + [(c_1 - c_2 - c_3 - 1)\dot{f}u_{xyt} - ((c_2 + c_3 - c_1)u_{xy} + c_2 x u_{xxy} + c_3 y u_{xyy} + c_4 z u_{xyz})\ddot{f} - c_5 u_{xxy}f^{(3)}]\partial_{u_{xyt}} \\
& + [(c_1 - c_2 - c_4 - 1)\dot{f}u_{zyt} - ((c_2 + c_4 - c_1)u_{xz} + c_2 x u_{xxz} + c_3 y u_{xyz} + c_4 z u_{xzz})\ddot{f} - c_5 u_{xxz}f^{(3)}]\partial_{u_{xzt}} \\
& + \sum_{3 \leq n+m+r \leq k-1}[(c_1 - nc_2 - mc_3 - rc_4 - 1)\dot{f}u_{x^n y^m z^r t} - (c_2 x u_{x^{n+1} y^m z^r} + c_3 y u_{x^n y^{m+1} z^r} \\
& \qquad + c_4 z u_{x^n y^m z^{r+1}} + (-c_1 + nc_2 + mc_3 + rc_4)u_{x^n y^m z^r})\ddot{f} - c_5 u_{x^{n+1} y^m z^r}f^{(3)}]\partial_{u_{x^n y^m z^r t}} \\
& + \text{terms of higher order } t\text{-derivatives}.
\end{aligned} \tag{17}$$

From (17), we can get the characteristic equation

$$\frac{\mathrm{d}t}{f} = \frac{\mathrm{d}x}{\dot{f}c_2x + c_5\ddot{f}} = \frac{\mathrm{d}y}{\dot{f}c_3y} = \frac{\mathrm{d}z}{\dot{f}c_4z} = \frac{\mathrm{d}u}{\dot{f}c_1u + c_6xyz\ddot{f}} = \ldots = \frac{\mathrm{d}u_{x^iy^jz^mt^r}}{U^{x^iy^jz^mt^r}} = \ldots . \tag{18}$$

Substituting the extensions $U^{x^iy^jz^mt^r}$ of the vector V_1 into (18) and solving it, we can obtain all the elementary invariants of (18). Here are some special examples:

$$I_1 = xf^{-c_2} - c_5\dot{f},\ I_2 = yf^{-c_3},\ I_3 = zf^{-c_4}, \tag{19}$$

$$I_4 = uf^{-c_1} - c_6I_1I_2I_3\dot{f} - \frac{1}{2}c_5c_6I_2I_3\ddot{f},\ I_5 = u_xf^{c_2-c_1} - c_6I_2I_3\dot{f}, \tag{20}$$

$$I_6 = u_yf^{-c_1+c_3} - c_6I_1I_3\dot{f} - \frac{1}{2}c_5c_6I_3\dot{f}^2,\ I_7 = u_zf^{-c_1+c_4} - c_6I_1I_2\dot{f} - \frac{1}{2}c_5c_6I_2\dot{f}^2, \tag{21}$$

$$I_8 = u_tf^{1-c_1} + \dot{f}(-c_1I_4 + c_2I_1I_5 + c_3I_2I_6 + c_4I_3I_7) - (c_6I_1I_2I_3 - c_5I_5)(f\ddot{f} - \dot{f}^2),\ I_9 = u_{xx}f^{2c_2-c_1}, \tag{22}$$

$$I_{10} = u_{xy}f^{-c_1+c_2+c_3} - \dot{f}c_6I_3,\ I_{11} = u_{xz}f^{-c_1+c_2+c_4} - \dot{f}c_6I_2,\ I_{12} = u_{yy}f^{-c_1+2c_3}, \tag{23}$$

$$I_{13} = u_{yz}f^{-c_1+c_3+c_4} - c_6I_1\dot{f} - \frac{1}{2}c_5c_6\dot{f}^2,\ I_{14} = u_{zz}f^{2c_4-c_1}, \tag{24}$$

$$I_{15} = u_{xt}f^{c_2+1-c_1} + [(c_2 - c_1)I_5 + c_2I_1I_9 + c_3I_2I_{10} + c_4I_3I_{11}]\dot{f} - (c_6I_2I_3 - c_5I_9)(f\ddot{f} - \dot{f}^2), \tag{25}$$

$$I_{16} = u_{xyz}f^{c_2+c_3+c_4-c_1} - c_6\dot{f},\ J_n^x = u_{x^n}f^{-c_1+nc_2},\ J_n^y = u_{y^n}f^{-c_1+nc_3},\ J_n^z = u_{z^n}f^{-c_1+nc_4}, \tag{26}$$

$$J_{nmr} = u_{x^ny^mz^r}f^{-c_1+nc_2+mc_3+rc_4},\ (n + m + r \geq 3 \text{ except for } n = m = r = 1), \tag{27}$$

$$I_{17} = u_{xxt}f^{1-c_1+2c_2} + ((2c_2 - c_1)I_9 + c_2I_1J_3^x + c_3I_2J_{210} + c_4I_3J_{201})\dot{f} + c_5J_3^x(f\ddot{f} - \dot{f}^2), \tag{28}$$

$$I_{18} = u_{xxxt}f^{1-c_1+3c_2} + ((3c_2 - c_1)J_3^x + c_2I_1J_4^x + c_3I_2J_{310} + c_4I_3J_{301})\dot{f} + c_5J_4^x(f\ddot{f} - \dot{f}^2), \tag{29}$$

$$I_{19} = u_{xxyt}f^{1-c_1+2c_2+c_3} + ((2c_2 + c_3 - c_1)J_{210} + c_2I_1J_{310} + c_3I_2J_{220} + c_4I_3J_{211})\dot{f} + c_5J_{310}(f\ddot{f} - \dot{f}^2), \tag{30}$$

$$I_{20} = u_{xxzt}f^{1-c_1+3c_2+c_4} + ((3c_2 + c_4 - c_1)J_{201} + c_2I_1J_{301} + c_3I_2J_{211} + c_4I_3J_{202})\dot{f} + c_5J_{301}(f\ddot{f} - \dot{f}^2), \tag{31}$$

Substituting the invariants shown in (19) - (31) into the generalized invariant equation (15), we can get various (3 + 1)-dimensional models which possess the Virasoro-type Lie point symmetry algebra (1). In general, (15) is f-dependent. According to the general theory of the last section, a Virasoro integrable model with algebra (1) should be f-independent. However, it is very difficult to find all the possible f-independent invariant equations because the invariants listed in (19) - (31) are dependent on the function f in a very complicated way. Fortunately, it is still possible to selected some f-independent invariant equations from (15). Here we give out only some special examples:

(i) Selecting $A = \frac{2c_2-c_1}{c_6}, B = -\frac{c_3}{c_6}, C = -\frac{c_4}{c_6}, D = -\frac{3c_2-c_1}{c_6}$ and from the V_1 invariant equation

$$I_{17}J_4^x - I_{18}J_3^x + AI_9I_{16}J_4^x + BI_{11}J_4^xJ_{210} + CI_{10}J_4^xJ_{201} + DI_{16}J_3^xJ_3^x - BI_{11}J_3^xJ_{310} - CI_{10}J_3^xJ_{301} = 0 \tag{32}$$

we can obtain the f-independent equation

$$\begin{aligned} &c_6(u_{xxt}u_{xxxx} - u_{xxxt}u_{xxx}) + (2c_2 - c_1)u_{xx}u_{xxxx}u_{xyz} - (3c_2 - c_1)u_{xyz}u_{xxx}^2 \\ &\quad + c_3u_{xz}(u_{xxx}u_{xxxy} - u_{xxy}u_{xxxx}) + c_4u_{xy}(u_{xxx}u_{xxxz} - u_{xxz}u_{xxxx}) = 0, \end{aligned} \tag{33}$$

where c_1, c_3, c_4, and c_6 are arbitrary constants. The corresponding Virasoro-type symmetry is

$$\sigma = f(t)\partial_t + (c_2 x\dot{f} + c_5\ddot{f})\partial_x - c_3 y\dot{f}\partial_y + c_4 z\dot{f}\partial_z + (c_1 u\dot{f} + c_6 xyz\dddot{f})\partial_u \tag{34}$$

(ii) If we take $c_4 = 2 + c_1 - c_3, A = \frac{c_1+2-c_3}{c_6}, B = D = -\frac{c_3}{c_6}, C = \frac{c_3-c_1-2}{c_6}$, we obtain an f-independent equation

$$\begin{aligned} &c_6(u_{xxyt}u_{xxxz} - u_{xxzt}u_{xxxy}) + (2 + c_1 - c_3)u_{xyz}u_{xxy}u_{xxxz} - c_3 u_{xyz}u_{xxz}u_{xxxy} \\ &\quad + c_3 u_{xz}(u_{xxxy}u_{xxyz} - u_{xxxz}u_{xxyy}) + (c_3 - c_1 - 2)u_{xy}(u_{xxzz}u_{xxxy} - u_{xxyz}u_{xxxz}) = 0 \end{aligned} \tag{35}$$

from the invariant equation

$$\begin{aligned} &I_{19}J_{301} - I_{18}J_{310} + AI_9 I_{16}J_4^x + BI_{11}J_{301}J_{220} + CI_{10}J_{211}J_{220} + DI_{16}I_{201}J_{310} - BI_{11}J_{310}J_{211} \\ &\quad - CI_{10}J_{202}J_{310} = 0, \end{aligned} \tag{36}$$

where c_1, c_3 and c_6 are arbitrary constants.

4. The Painlevé Property of (3+1)-dimensional Virasoro Integrable Models

In this section, we would like to select some Painlevé integrable models from the Virasoro integrable models listed in the last section. The singularity analysis formulated by Weiss, Tabor, and Carnevale (WTC) [10] is a useful and simple method to check the Painlevé integrability of a model. According to the WTC approach, we say a model possesses the Painlevé property if all the solutions of the model are single-valued about an arbitrary singularity manifold which is given by $\phi(x, y, z, t) = 0$. For simplicity to prove the Painlevé property, Kruskal has proposed that the singular manifold $\phi(x, y, z, t)$ can be replaced by $x + \phi_1(y, z, t)$ with arbitrary analytical $\phi_1(y, z, t)$ [11]. In order to perform the Painlevé analysis of the (3+1)-dimensional PDE model (33), we can rewrite it as

$$\begin{aligned} &(v_{xt}v_{xxx} - v_{xxt}v_{xx}) + \frac{3}{2}v_x v_{xxx}v_{yz} - 2v_{yz}v_{xx}^2 \\ &\quad + c_3 v_z(v_{xx}v_{xxy} - v_{xy}v_{xxx}) \\ &\quad + c_4 v_y(v_{xx}v_{xxz} - v_{xz}v_{xxx}) = 0 \end{aligned} \tag{37}$$

and fix $c_1 = -1/2$, $c_2 = 1/2$, $c_5 = 0, and\ c_3 + c_4 = 0$ by using $v = u_x$.

With the help of the leading order analysis that is used in the standard WTC method, v can be expanded about the singularity manifold $\phi = x + \phi_1(y,\ z,\ t)$ as

$$v = \sum_{j=0}^{\infty} v_j \phi^{j-2}, \tag{38}$$

where $\phi_1 = \phi_1(y, z, t)$ and $v_j = v_j(y, z, t)$ are analytical functions of y, z, t. Substituting (38) into (37), we get the recursion relation of the coefficients v_j

$$j(j+1)(j-2)v_j = F_j(\phi_1, \phi_{1y}, \ldots, v_0, v_1, \ldots, v_{j-1}), \quad (j = 0, 1, 2, \ldots), \tag{39}$$

where $F_j(\phi_1, \phi_{1y}, \ldots, v_0, v_1, \ldots, v_{j-1})$ is a complicated function of $\phi_1, \phi_{1y}, \ldots, v_0, v_1, \ldots, v_{j-1}$.

From (39) we know that the resonances occur at

$$j = -1, 0, 2. \tag{40}$$

The resonance at $j = -1$ corresponds to the arbitrary singularity manifold ϕ and j=0 corresponds to the arbitrary function v_0. From the recursion relation (39), we have

$$j = 1,\ v_1 = \frac{\frac{2}{3}(c_3\phi_{1z}v_{0y} + c_4\phi_{1y}v_{0z})}{\phi_{1z}\phi_{1y}}, \tag{41}$$

$$j = 2, \tag{42}$$

$$\begin{aligned} &2c_3 v_{0z}v_{oy} - 2\phi_{1yz}v_0 v_1 - 4c_4\phi_{1y}v_{1z}v_0 - 4c_3\phi_{1z}v_{1y}v_0 \\ &-2v_1\phi_{1y}v_{0z} + 3\phi_{1z}\phi_{1y}v_1^2 - 2v_1\phi_{1z}v_{0y} + 2c_4 v_{0z}v_{0y} \\ &\quad - 2c_4 v_1\phi_{1y}v_{0z} - 2c_3 v_1\phi_{1z}v_{0y} = 0. \end{aligned}$$

According to the standard WTC approach [10], we know that if a model possesses the Painlevé property, all the resonance conditions should be satisfied identically. So, for the model (33), if it is Painlevé integrable, a further compatibility condition (42) must be satisfied. It is clear that (42) is satisfied identically only for $c_3 = -c_4 = 0$. So (33) with $c_3 = c_4 = 0$ is integrable under the meaning that it possesses the Painlevé property.

Using the similar analysis to the model (35), we find that it is only Virasoro integrable but not Painlevé integrable.

5. Summary and Discussion

In summary, starting from every realization of the Virasoro-type of symmetry algebra (1) we are able to obtain various Virasoro integrable models. Using two special types of concrete realization of the Virasoro symmetry algebra, we have write down four special Virasoro integrable models. Usually, a Virasoro integrable model might not be integrable under other meanings. Fortunately, we have shown that some types of Virasoro integrable models may also be Painlevé integrable, (33) for $c_1 = -1/2, c_2 = 1/2, c_3 = c_4 = 0$.

Actually, from every realization of the Virasoro-type of symmetry algebra (1) we may obtain *infinitely many* Virasoro integrable models. For instance, some terms like

$$(u_{x^m})^a(u_{y^n})^b(u_{z^p})^c, (u_{x^{m_1}y^{n_1}})^{a_1}(u_{z^{p_1}})^{b_1}, \quad (u_{x^{m_2}y^{n_2}z^{p_3}})^{a_2}, \ldots \tag{43}$$

for the arbitrary real constants a, b, c, a_1, b_1 and a_2 with the conditions

$$a(mc_2 - c_1) + b(nc_3 - c_1) + c(pc_4 - c_1) = 6c_2 - 2c_1 + 1, \quad m, n, p \geq 2, \tag{44}$$

$$a_1(m_1c_2 + n_1c_3 - c_1) + b_1(p_1c_4 - c_1) = 6c_2 - 2c_1 + 1, \quad p_1 \geq 2,\ m_1 + n_1 \geq 3, \tag{45}$$

$$a_2(m_2c_2 + n_2c_3 + p_2c_4 - c_1) = 6c_2 - 2c_1 + 1, \quad m_2 + n_2 + p_2 \geq 3,\ m_2n_2p_2 \neq 1 \tag{46}$$

can also be added to (33).

Though we have obtained many (3+1)-dimensional Virasoro integrable equations and one model is Painlevé integrable, there are various open, interesting and important questions. For instance, what kinds of Virasoro integrable model would be Painlevé integrable at the same time? Can we find, some (3+1)-dimensional IST integrable or Lax integrable models from the Virasoro integrable models? Can we find and how to find the multisoliton solutions of the Virasoro integrable models? All these problems are worthy of further study.

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