Research Note on New Similarity Reductions of a Variable-Coefficient Korteweg-de Vries Equation

Bo Tian
Department of Applied Mathematics and Physics, Beijing University of Aeronautics and Astronautics, Beijing 100083, China
Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing 100088, China

Z. Naturforsch. 52a, 463–464 (1997); received February 7, 1997

For a variable-coefficient Korteweg-de Vries equation we obtain 4 new similarity reductions to the Painlevé type equations or the Weierstrass elliptic function equation.

In this note, we report the application of the Clarkson-Kruskal direct method [1], which is powerful for the constant-coefficient nonlinear evolution equations, to a variable-coefficient Korteweg-de Vries equation [2],

\[ u_t + f(u)u_x + g(u)u_{xxx} = 0, \]

where \( f(t) \neq 0 \) and \( g(t) \neq 0 \).

Firstly, it is sufficient to seek a similarity reduction of (1) in the form

\[ u(x,t) = a(x,t) + b(x,t)w[z(x,t)], \]

where the functions \( a(x,t), b(x,t), z(x,t) \) and \( w(z) \) are to be determined, rather than the more general form of \( u(x,t) = U[x,t,w[z(x,t)]] \) (proof ignored). Also, we will consider the interesting case of \( z(x,t) \neq 0 \).

After the substitution of (2), we impose the condition that (1) be an ordinary differential equation for \( w(z) \), i.e., demand the ratios of the coefficients of different derivatives and powers of \( w(z) \) to be functions of \( z \) only. Computerized symbolic computation thus leads to a set of conditions for \( a(x,t), b(x,t) \) and \( z(x,t) \), based on which we find the following format of similarity reductions:

\[ u(x,t) = -\frac{\theta(t)}{f(t)} x + \phi(t), \]

with \( z(x,t) = \theta(t) x + \phi(t), \)

where the functions \( \theta(t) \neq 0, \phi(t), w(z) \), as well as the constraints on \( f(t) \) and \( g(t) \), are given below case by case:

**Similarity reduction I:**

\[ \theta(t) = k, \]

\[ \phi(t) = \begin{cases} C k^6 \int f(t) \frac{g(t)^2}{f(t)} \, dt \quad \text{when } C \neq 0, \\ M \int f(t) \, dt \quad \text{otherwise} \end{cases} \]

where \( C, k \neq 0 \) and \( M \) are arbitrary constants, while \( w(z) \) satisfies the ordinary differential equation

\[ w'''' + w w' + C = 0, \]

which is equivalent to either the first Painlevé equation as \( C \neq 0 \) or the Weierstrass elliptic function equation as \( C = 0 \).

The constraint on \( f(t) \) and \( g(t) \) is

\[ A \left[ \frac{g(t)}{f(t)} \right] = 0, \quad \text{where } A = \frac{d}{dt}. \]

**Similarity reduction II:**

\[ \theta(t) = E \frac{f(t)}{g(t)}, \]

while \( \phi(t) \) satisfies

\[ \frac{f(t)}{g^2(t)} \left[ \frac{f'(t)}{f(t)} \frac{\phi'(t)}{\theta(t)} + \frac{2\theta(t)'(t)}{f(t)} \frac{\phi'(t)}{\theta(t)} \phi''(t) \right] = B \phi(t) + D, \]

where \( D \) and \( E \neq 0 \) are arbitrary constants, while \( B \) is a constant given by (13) below, and \( \theta(t) \) by (9). \( w(z) \) must satisfy

\[ w'''' + w w' + B z + D = 0, \]

which is equivalent to the first Painlevé equation when \( B = 0 \), or else, not of the Painlevé type.

The constraints on \( f(t) \) and \( g(t) \) are

\[ A \left[ \frac{g(t)}{f(t)} \right] = 0, \]

\[ \frac{f(t)}{g^2(t)} \left[ \frac{f'(t)}{f(t)} \frac{\theta'(t)}{\theta(t)} + \frac{2\theta(t)'(t)}{f(t)} \frac{\theta'(t)}{\theta(t)} \right] = B = \text{constant}, \]

with \( \theta(t) \) given by (9).
Similarity reduction III:
\[
\theta(t) = -3A \int g(t) \, dt \left[ \frac{H}{2A^2} - 3A \int g(t) \, dt \right]^{-1/3},
\]
\[
\phi(t) = H - 2 \left[ \frac{H}{2A^2} - 3A \int g(t) \, dt \right]^{-1/3},
\]
where \( A \neq 0 \) and \( H \) are arbitrary constants, while \( w(z) \) satisfies
\[
w'' + w' + A w - 2A^2 z + H = 0,
\]
the solutions of which are known to be related through a one-to-one transformation to those of the second Painlevé equation.

The constraint on \( f(t) \) and \( g(t) \) is
\[
A \left[ \frac{g(t)}{f(t)} \right] = 0, \quad \text{where} \quad A = \frac{d}{dt} \ln.
\]

\[ \text{(17)} \]

Similarity reduction IV:
\[
\theta(t) = -\frac{f(t)}{g(t)} \left[ 3N \left[ \frac{f^3(t)}{g^2(t)} \right] \right]^{-1/3},
\]
\[ \text{(18)} \]

\[ \phi(t) \begin{cases} 
\text{satisfies} & \frac{f'(t)}{f^2(t)} \frac{\phi'(t)}{\theta(t)} + \frac{2\theta'(t)}{f(t)} \frac{\phi'(t)}{\theta'(t)} - \frac{\phi''(t)}{f(t)} \frac{\theta'(t)}{\theta(t)} = \frac{P}{f(t)} \frac{g^2(t)}{f(t)} \frac{\theta^5(t)}{g(t)} & \text{if} \quad P \neq 0 \quad \& \quad R = 0, \\
= \theta(t) & \text{if} \quad P = 0 \quad \& \quad R = 0, \\
= \theta(t) - P/R & \text{otherwise}
\end{cases} \]

\[ \text{(19)} \]

where \( N \neq 0 \) and \( P \) are arbitrary constants, while \( R \) is a constant given by (22) below, and \( \theta(t) \) by (18). \( w(z) \) satisfies
\[
w'' + w' + N w + R z + P = 0,
\]
which is a form of the second Painlevé equation only when \( R + 2N^2 = 0 \).

The constraints on \( f(t) \) and \( g(t) \) are
\[
A \left[ \frac{g(t)}{f(t)} \right] \neq 0,
\]
\[ \text{(21)} \]

\[
\frac{R}{N^2} = -2 + 9 \frac{g^4(t)}{f^2(t) g^2(t)} \left[ \frac{f^3(t)}{g^2(t)} \int \right]^2 \frac{f^3(t)}{g^2(t)} - 3 \frac{f(t)}{f^2(t)} \frac{g(t)}{g(t)} - 3 \frac{f'(t)}{f^2(t)} \frac{g'(t)}{g(t)} + \frac{g''(t)}{f(t) g(t)},
\]
with \( \theta(t) \) given by (18).

Acknowledgement

This work has been supported by the Outstanding Young Faculty Fellowship and the Research Grants for the Scholars Returning from Abroad, State Education Commission of China.