

On Hide's Magnetic Analogue of Ertel's Vorticity Theorem

W. Schröder^a and H.-J. Treder^b

^a Hechelstr. 8, D-28777 Bremen-Roennenbeck
^b Rosa-Luxemburg-Str. 17a, D-14482 Potsdam

Z. Naturforsch. **52a**, 210–211 (1997);
received January 8, 1997

The relativistic formulation of Hide's "magnetic analogue" of Ertel's potential vorticity theorem is Dirac's "new classical theory of electrons".

Key words: Hide's and Ertel's theorems; Potential vorticity; Potential magnetic field; Geomagnetism.

Dirac's "new classical theory of electrons" [1, 2] describes the relativistic hydrodynamics of conductive media with an "Ohm law"

$$u^\mu = \frac{e}{mc} (A^\mu - g^{\mu\nu} \partial_\nu S), \quad (\mu, \nu = 0, 1, 2, 3). \quad (1)$$

In (1), $u^\mu = \frac{dx^\mu}{d\tau}$ is the four-dimensional velocity, $d\tau$ the differential of the proper time, A^μ the four-dimensional electro-magnetic vector potential, and S a gauge function. $g_{\mu\nu}$ means the metrical Minkowski tensor, and e and m are the charge and the rest mass of an electron, respectively.

According to (1), the norm of the velocity is

$$-c^2 = u_\mu u^\mu = \frac{e^2}{m^2 c^2} (A_\mu A^\mu + g^{\mu\nu} \partial_\mu S \partial_\nu S - 2 A^\mu \partial_\mu S). \quad (2)$$

The tensor of the four-dimensional rotation is

$$\omega_{\mu\nu} = \partial_\nu u_\mu - \partial_\mu u_\nu = \frac{e}{mc} (\partial_\nu A_\mu - \partial_\mu A_\nu) = \frac{e}{mc} F_{\mu\nu} = -\omega_{\nu\mu}, \quad (3)$$

where $F_{\mu\nu}$ is the anti-symmetric Maxwellian tensor of the electro-magnetic field.

The four-dimensional relativistic generalization of Helmholtz' theorem of the conservation of vorticity

Reprint requests to W. Schröder.

0932-0784 / 97 / 0100-0210 \$ 06.00 © – Verlag der Zeitschrift für Naturforschung, D-72072 Tübingen

means [3–5]

$$d\omega_{\alpha\beta} = u^\lambda \partial_\lambda \omega_{\alpha\beta} = \omega^\lambda_\alpha \partial_\beta u_\lambda - \omega^\lambda_\beta \partial_\alpha u_\lambda + \partial_\beta \frac{du_\alpha}{d\tau} - \partial_\alpha \frac{du_\beta}{d\tau}, \quad (4)$$

where $\frac{d\Phi}{d\tau} = u^\lambda \partial_\lambda \Phi$.

Because of the identity $\frac{d}{d\tau} dx^\alpha = u^\lambda \partial_\lambda dx^\alpha$ one can write (4), with the help of the Cartan exterior differential [6]

$$[dx^\alpha \wedge dx^\beta] = -[dx^\beta \wedge dx^\alpha],$$

like a relativistic formulation of Ertel's vorticity theorem [7]; see also [8]

$$\frac{d}{d\tau} (\omega_{\mu\nu} [dx^\mu \wedge dx^\nu]) = 2 \partial_\nu \frac{du_\mu}{d\tau} [dx^\mu \wedge dx^\nu]. \quad (5)$$

By the Cartan-Stokes integral theorem, (5) yields the circulation theorem [6]

$$\frac{d}{d\tau} \int_{C_2} \omega_{\mu\nu} [dx^\mu \wedge dx^\nu] = 2 \oint_{C_1} \frac{du_\mu}{d\tau} dx^\mu \quad (6)$$

(with $C_1 = \partial C_2$).

According to Poincaré's rules of exterior differentiation, (5, 6) with Dirac's law (1) yield an analogous equation for the electromagnetic field strength $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$, namely the non-holonomic expression

$$\frac{d}{d\tau} (F_{\mu\nu} [A^\mu \wedge A^\nu]) = 2 \partial_\nu \frac{dA_\mu}{d\tau} [A^\mu \wedge A^\nu], \quad (7)$$

together with a circulation theorem of the electromagnetic field

$$\frac{d}{d\tau} \int_{C_2} F_{\mu\nu} [dx^\mu \wedge dx^\nu] = 2 \oint_{C_1} \frac{dA_\mu}{d\tau} dx^\mu \quad (8)$$

according to the holonomic equations

$$\frac{d}{d\tau} (F_{\mu\nu} [dx^\mu \wedge dx^\nu]) = 2 \partial_\nu \frac{dA_\mu}{d\tau} [dx^\mu \wedge dx^\nu]. \quad (9)$$

These equations are a relativistic generalization of Hide's magnetic analogue [9, 10] of Ertel's vorticity theorem. The relativistic point of view generalizes the "magnetic analogue" of Hide to an "electro-magnetic" analogue.



Dieses Werk wurde im Jahr 2013 vom Verlag Zeitschrift für Naturforschung in Zusammenarbeit mit der Max-Planck-Gesellschaft zur Förderung der Wissenschaften e.V. digitalisiert und unter folgender Lizenz veröffentlicht: Creative Commons Namensnennung-Keine Bearbeitung 3.0 Deutschland Lizenz.

Zum 01.01.2015 ist eine Anpassung der Lizenzbedingungen (Entfall der Creative Commons Lizenzbedingung „Keine Bearbeitung“) beabsichtigt, um eine Nachnutzung auch im Rahmen zukünftiger wissenschaftlicher Nutzungsformen zu ermöglichen.

This work has been digitized and published in 2013 by Verlag Zeitschrift für Naturforschung in cooperation with the Max Planck Society for the Advancement of Science under a Creative Commons Attribution-NoDerivs 3.0 Germany License.

On 01.01.2015 it is planned to change the License Conditions (the removal of the Creative Commons License condition "no derivative works"). This is to allow reuse in the area of future scientific usage.

- [1] P. A. M. Dirac, Proc. Roy. Soc. London **A 209**, 291 (1951).
- [2] P. A. M. Dirac, Proc. Roy. Soc. London **A 212**, 330 (1951).
- [3] H. Ertel, Ein System von Identitäten und seine Anwendung zur Transformation von Wirbelgleichungen der Hydrodynamik, Monatsberichte Deutsche Akad. Wiss. Berlin **4**, 292 (1962).
- [4] H. Ertel, Analogia entre las ecuaciones del movimiento y las ecuaciones del torbellino en la Hidrodinamica. Gerlands Beitr. Geophysik **72**, 312 (1963).
- [5] H.-J. Treder, Die allgemein-kovariante, relativistische Verallgemeinerung des 1. Helmholtzschen Wirbelsatzes, Gerlands Beiträge Geophysik **78**, 436 (1969).
- [6] A. Lichnerowicz, Relativistic hydrodynamics and magnetohydrodynamic, New York, 1967.
- [7] H. Ertel, Meteorol. Zs. **59**, 227 (1942) (engl. transl. in W. Schröder ed., Geophysical Hydrodynamics and Ertel's Potential Vorticity, Bremen 1991).
- [8] H.-J. Treder, Zur allgemein-relativistischen Integralform der Helmholtzschen Wirbeltheoreme, Gerlands Beiträge Geophysik **79**, 1 (1970).
- [9] R. Hide, The magnetic analogue of Ertel's vorticity theorem, Ann. Geophys. **1**, 59 (1983).
- [10] R. Hide, Potential magnetic field and potential vorticity in magnetohydrodynamics, Geophys. Int. J. **125**, F1–F3 (1996).

Nachdruck — auch auszugsweise — nur mit schriftlicher Genehmigung des Verlages gestattet
Verantwortlich für den Inhalt: A. KLEMM
Satz und Druck: Konrad Tritsch, Würzburg